# CK-12

# Trigonometry Teacher's Edition
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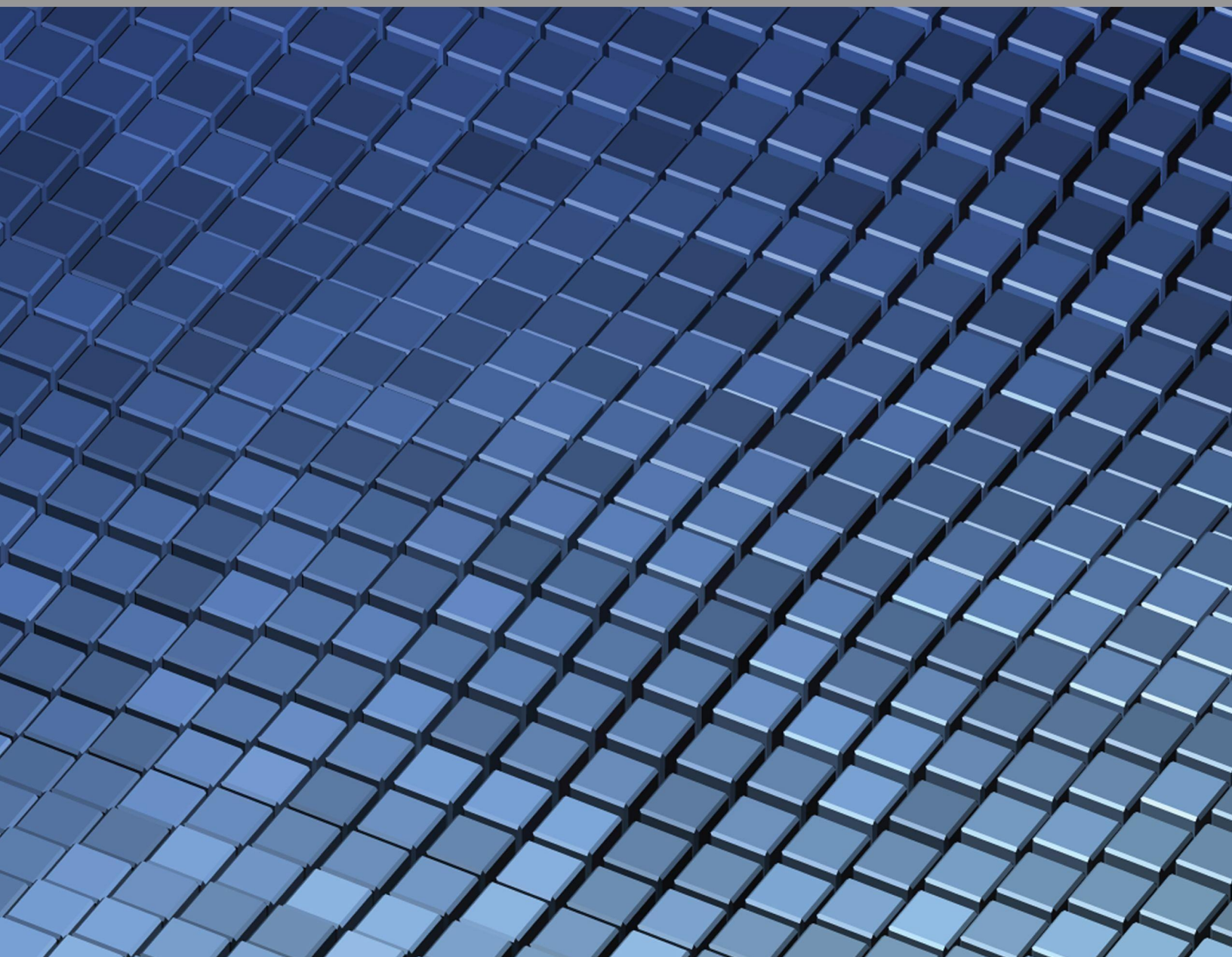
# Trigonometry Teacher's Edition

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# CHAPTER 1

# Trigonometry TE - Teaching Tips

# 1.1  Trigonometry and Right Angles

This Trigonometry Teaching Tips FlexBook is one of seven Teacher's Edition FlexBooks that accompany the CK-12 Foundation's Trigonometry Student Edition.

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## Basic Functions

**The Basics of Functions**

Some students' eyes may glaze over at the equation used in the "gas mileage" situation, especially if it's been a while since their last algebra class. Watch for students who look confused and walk them through setting up the equation if necessary.

Another way to explain the definition of **domain** is that it is the set of real $x-$values you can plug into the function that will produce a real number as the output. You can show that the domain of $y = 3x$ is the set of all real numbers because any real number $x$ can be multiplied by 3 to get a real number $y$; then you can show that the domain of $y = \sqrt{x}$ is restricted to the nonnegative real numbers by demonstrating that you get a real value for $y$ when you plug in a positive or zero value for $x$, but not when you plug in a negative value.

Similarly, you can explain that the **range** of a function is the set of $y-$values that could be the possible outputs of the function given a real number $x$ as the input.

The "rounding" example provides a good opportunity to remind students that rounding "up," in the case of a negative number, means rounding *toward* zero rather than *away from* zero; since $-5$ is *greater than* $-6$, $-5.5$ should be rounded up to $-5$.

**Families of Functions**

In the "families of functions" table, you may need to clarify what "these functions have a highest exponent of 2" means—i.e., the highest power that appears anywhere in the equation is 2. (We can also say that the degree of the function is 2, or that the function is of the "second degree.") The same holds, of course, for "these functions have a highest exponent of 3," but you also may want to clarify "the ends of the graph have opposite behavior": it simply means that one end of the graph goes up while the other goes down. You may also want to graph $y = x^3$ to show why it has no local maximum or minimum.

And finally, for students who may have forgotten how asymptotes work, it's worth reminding them that the values of the function approach the asymptote but never actually reach it. Specifically, if a function has a horizontal asymptote, it means that y will get closer and closer to that value as $x$ approaches infinity, but will never quite reach it; if a function has a vertical asymptote, it means that $y$ will get closer and closer to infinity as $x$ gets closer to the given value, but the function is undefined when $x$ is exactly equal to that value.

A useful way to explain direct and inverse variation is that with direct variation, the dependent variable increases when the independent variable increases, while with inverse variation, the dependent variable decreases when the independent variable increases. This makes the contrast between the two types of function clearer.

You can clarify the definition of a periodic function by explaining that all the values of the function repeat themselves every $p$ units. It may be useful to demonstrate with the "weather" example above: $p$ in this case is 12, so you can show that $f(14) = f(2), f(15) = f(3)$, and so on.

**Points to Consider**

(You may want to go over these as a group each lesson.)

Using a calculator to graph functions is quicker and more accurate than doing it by hand, but it can be hard to see precisely where the important points on the graph are.

# Angles in Triangles

### Similar Triangles

It may be useful to note that the proportions that show that the ratios of corresponding sides are equal can be derived directly from the proportions that show that the side ratios within the two triangles are equal. (For example, $\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{BC}}{\overline{EF}}$ can be derived directly from $\frac{\overline{AB}}{\overline{BC}} = \frac{\overline{DE}}{\overline{EF}}$.)

If you want to move through example 4 a little more quickly, you can point out that the side lengths in the second triangle are simply half those in the first.

Students may ask whether ASA and SAA are also criteria for determining if two triangles are similar, since criteria like those exist for determining if triangles are congruent. Explain that they are, but for a very simple reason: if two of the angles are congruent, then the third angle must also be congruent, and so the ASA and SAA cases simply reduce to the AAA case.

The HL case, on the other hand, is a special example of SSA: two of the sides are proportional, and an angle that is not between them is congruent. Normally, this would not be enough to determine that two triangles are similar, but what helps us here is that we are dealing with right triangles, which means that we can use two side lengths to determine the third. Once we've found that the other leg is proportional too, then instead of looking at this as a case of SSA, we can see it as a case of SSS: all three pairs of sides are proportional. (Incidentally, we could also see it as a case of SAS: the two legs are proportional, and the angle between them (the right angle) is congruent.)

**Points to Consider**

The answer to question #1 can be demonstrated in more than one way. First, you can draw two right angles with the included side between them, and show that the other two sides are now parallel, meaning that they can't ever meet to make a triangle.
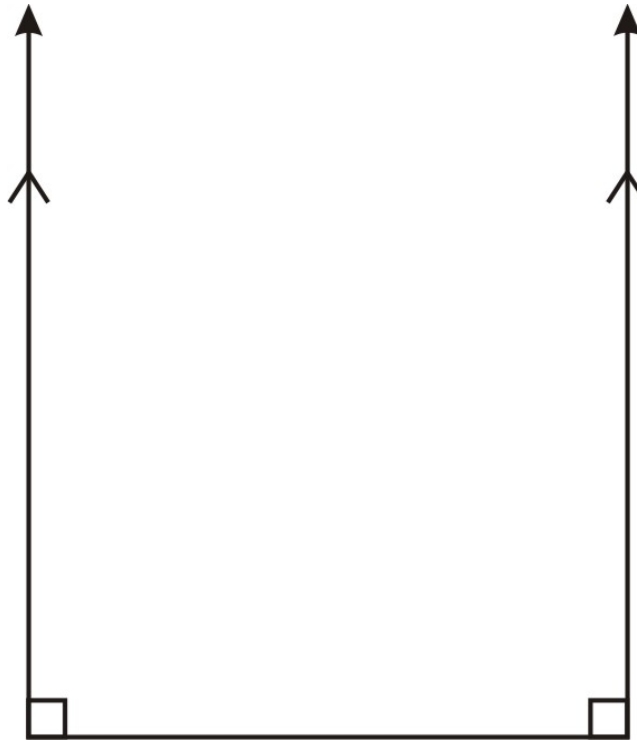
Second, you can point out that two right angles add up to 180°, and since that is the sum of all three angles of a triangle, that would mean the third angle would have to measure 0°, which is not possible.

Similar reasoning holds for question #2. Drawing two obtuse angles with the included side between them demonstrates even more clearly that the other two sides could never meet, and adding together two angle measures greater than 90° each would give you a sum greater than 180°, which is impossible even before you consider the measure of the third angle.

Question #3 has a different answer depending on what situation you are considering. If you just look at an angle in isolation, then in a sense it cannot have a measure greater than 180°, because an angle of, say, 200° could just as easily be described as an angle of 160° viewed from the other side. However, when you are measuring angles of rotation, as the next lesson will cover, an angle can measure more than 180º or even more than 360°. Mentioning this might be a good segue to the next lesson.

# Measuring Rotation

**Measuring Angles**

Some students may need to see example 3 worked out in more detail. You may need to spell out that the circumference of the smaller wheel is $\pi$ meters and that of the larger is $2\pi$ meters; then, rather than simply explaining that the larger wheel rotates once every time the smaller one rotates twice (and therefore rotates twice when the small one rotates four times), you may need to show that the four rotations of the smaller wheel cause its circumference to travel 4 meters along the larger wheel, and that this is equal to $\frac{4\pi}{2\pi}$ or 2 rotations of the larger wheel.

**Angles of Rotation in Standard Position**

Students may not have encountered the terms "initial side" and "terminal side" before. Explain, if necessary, that these terms are specific to this particular situation; when we place an angle in standard position, the initial side is just what we call the side we chose to place along the $x-$axis, and the terminal side is simply the other side.

**Co-terminal Angles**

*1.1. TRIGONOMETRY AND RIGHT ANGLES*

In working through the next example, you may want to take a moment to remind students which quadrant is which (quadrants I through IV proceed counterclockwise starting from the upper right). Knowing the quadrants will be important in upcoming lessons.

Another way to generate the angle $-315°$, of course, is to subtract $315°$ from $360°$ to get $45°$. This method can be faster than rotating clockwise, but students should familiarize themselves with both techniques.

**Points to Consider**

Real-life instances of angles of rotation might include a wheel, a swinging door, a doorknob, or a screw.

**Review Questions**

You may need to walk students through problem 7. First they must find the total distance the car's inner wheel travels, which is a quarter (90 degrees [U+0080] [U+0099] worth) of the circumference of a circle with a 100m radius. Then they must find the number of rotations the inner wheel makes in traveling that distance, which takes two steps: first find the circumference of the wheel based on the given diameter of .6m, and then divide the total distance traveled by the circumference of the wheel to find the number of rotations it makes. Next, they must find the distance the outer wheel travels. Since the wheels are 2m apart, the outer wheel follows a curve with radius 2m greater than the curve the inner wheel follows, so the distance it travels is a quarter of the circumference of a circle with a 102m radius. Then, dividing that distance by the circumference of the wheel (already found) gives the number of rotations the outer wheel makes. Finally, they must subtract to find how many more rotations the outer wheel makes than the inner.

# Defining Trigonometric Functions

### The Sine, Cosine, and Tangent Functions

Another way to explain the domain and range of the first three trigonometric functions is as follows: The trigonometric functions take angles as their input, and their output consists of particular ratios of side lengths.

The mnemonic SOH CAH TOA (Sin: Opposite/Hypotenuse; Cosine: Adjacent/Hypotenuse; Tangent: Opposite/Adjacent—pronc roughly "soak a toe-a") may help students remember the ratios. Another mnemonic is "SCoTt, Oscar Has A Heap Of Apples"—that is, for Sin, Cos, and Tan respectively, the ratios are $\frac{O}{H}, \frac{A}{H}$, and $\frac{O}{A}$.

It may be worth stressing that $\sin(x), \cos(x)$, and $\tan(x)$ are abbreviations for types of functions, and do not indicate that anything is being multiplied.

### Secant, Cosecant, and Cotangent Functions

The fact that the secant, cosecant, and cotangent are reciprocals of the cosine, sine, and tangent functions respectively will be made explicit in a later section of the text, but it may be useful to point it out now, as this may make it easier for students to remember those ratios.

### Trigonometric Functions of Angles in Standard Position

In example 4, you may need to clarify that although the two legs of the triangle in the diagram are labeled 3 and 4, the $x-$coordinate we are working with is actually $-3$, and so when finding the values of the trig functions, we must plug in $-3$ as the length of that leg. (This is where trig functions of angles of rotation start to differ from trig functions of angles in right triangles.)

### Points to Consider

The Pythagorean Theorem is useful in trigonometry in at least two ways: it helps us find the third side of a right triangle when we need to, and it helps establish some important trigonometric identities. However, the latter won't be covered for a couple more lessons, so you may or may not want to even mention it at this point.

Values of trig functions can be negative when we are dealing with angles of rotation instead of angles in right

triangles, because we define the functions in a slightly different way to allow us to describe many more cases. Angles in right triangles must be less than $90°$, and when we work out the trig functions for those angles, we always get positive numbers because the triangles' side lengths are always positive. But when we define the trig functions by reference to $x-$ and $y-$coordinates and the unit circle, we now have a way of finding their values for angles greater than $90°-$and it turns out that some of those values are negative, because the $x-$ and $y-$coordinates we use to find them are sometimes negative. Similarly, trig values can be undefined when we try to find them for quadrantal angles, because some of the coordinates of those angles equal zero.

(All of this will be covered in more detail in the next lesson.)

The unit circle is useful because it gives us an easy way to calculate the trig functions for any given angle; then, because of similar triangles, we know that those values will be the same when we see that same angle in any triangle, even if the hypotenuse of the triangle is not 1. For example, the unit circle tells us that the cosine of $50°$ is about 0.6428, so whenever we see a right triangle with a $50°$ angle in it, we know (because the triangles are similar) that the ratio of the adjacent leg to the hypotenuse will always be 0.6428, without having to measure the sides.
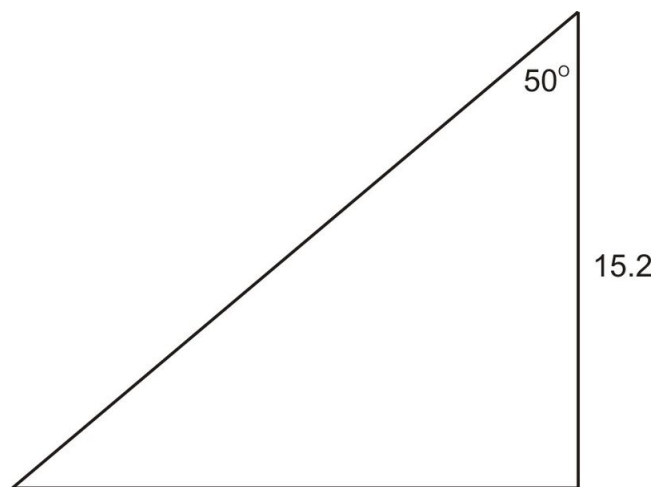


$$\frac{\text{adjacent}}{\text{hypotenuse}} = 0.6428$$

$$\text{adjacent} = \text{hypotenuse} * 0.6428$$

$$15.2 = \text{hypotenuse} * 0.6428$$

$$\text{hypotenuse} = \frac{15.2}{0.6428} \approx 23.6465$$

## Trigonometric Functions of Any Angle

**Reference Angles and Angles in the Unit Circle**

After example 1, you may need to show more explicitly how we know the ordered pair for $150°$ based on the ordered pair for $30°$. Remind students that a $150°$ angle is the reflection across the $y-$axis of a $30°$ angle (refer to the earlier diagram), and remind them (and demonstrate visually) that when we reflect a point across the $y-$axis, its $y-$coordinate stays the same and its $x-$coordinate changes sign.

Example 2 provides another opportunity to make this clear. Each angle that has $60°$ as its reference angle is simply the angle we get if we reflect a $60°$ angle across the $x-$axis (putting it in the fourth quadrant), the $y-$axis (putting it in the second quadrant), or both (putting it in the third quadrant). If we reflect it across the $x-$axis, its $x-$ coordinate

stays the same and its $y-$coordinate changes sign; if we reflect it across the $y-$axis, its $y-$coordinate stays the same and its $x-$coordinate changes sign; and if we do both, both coordinates change sign. So we can easily find the coordinates for any angle once we know its reference angle and which quadrant it is in.

**Trigonometric Function Values in Tables**

When you arrive at the table of trig function values, you may want to encourage students to compare the values of the trig functions for pairs of supplementary angles (like $85°$ and $95°$, or $125°$ and $55°$). The table makes it clear that cosines of supplementary angles are equal, and sines and tangents of supplementary angles are opposites. Thinking in terms of reference angles will make it clearer why this happens: an angle between $90°$ and $180°$ has a reference angle that is equal to its supplement, so the values of the trig functions for that angle are closely related to the values for its supplementary angle.

Because of this fact, there is another way to find the answer to example 6a; challenge students to figure out what it is. (Hint: what is the reference angle of $130°$?)

You may also want to encourage students to compare the sine and cosine values for pairs of complementary angles, like $35°$ and $55°$; the table shows that the sine of an angle is equal to the cosine of its complement. This fact will be useful later, and the reason for it will be clearer when we study the unit circle in more detail.

**Points to Consider**

Here's one way to explain the difference between the measure of an angle and its reference angle: when you start at the positive $x-$axis and rotate counterclockwise to get to the terminal side of the angle, the distance you've traveled is the angle measure. When you start at the terminal side of the angle and travel by the quickest route to the closest portion of the $x-$axis, the distance you've traveled is the reference angle. (Demonstrate this visually with at least one angle that is not in the first quadrant. For example, with a $240°$ angle, you can show that $240°$ is the clockwise distance from the positive $x-$axis, but $60°$ is the shortest distance to the closest part of the $x-$axis.)

An angle is the same as its reference angle only when it is between $0°$ and $90°$.

The simplest way to answer question #2 is by considering how we find values of trig functions on the unit circle. The values of sine and cosine there are simply equal to the $y-$coordinate and $x-$coordinate, respectively, of the ordered pair that defines the angle, so the angles that have the same (or opposite) sine (or cosine) value will simply be the ones with the same (or opposite) $y-$coordinate (or $x-$coordinate).

**Review Questions**

The function in problem 12 is fairly complex (although it can be simplified) and it shouldn't be immediately obvious what the graph will look like. Students should be able to simplify the expression under the square root sign based on their conjecture from the previous problem; after that, the best they can do is figure out what the function's values will be for a few key angles ($30°, 45°, 60°$, and so on), plot the points, and sketch a graph based on those points, and then graph the function on a calculator to compare it with their sketch.

# Relating Trigonometric Functions

**Reciprocal Identities**

After explaining that we obtain the reciprocal of a fraction by flipping the fraction, you might need to clarify how to obtain the reciprocal of something that is not a fraction. For example, the reciprocal of $x$ is $\frac{1}{x}$, and that's because $x$ is equivalent to the fraction $\frac{x}{1}$, which we can flip to find the reciprocal. Similarly, a trig value like $\cos \theta$ is equivalent to $\frac{\cos \theta}{1}$, so if $\sec \theta$ is the reciprocal of $\cos \theta$, that means it is equivalent to $\frac{1}{\cos \theta}$.

The identity $1 = \sin^2 x + \cos^2 x$ may not be immediately obvious; it hasn't previously been explicitly mentioned, although students were encouraged to discover it for themselves in Review Question 11 of the previous lesson. If you wish, you can demonstrate how to derive it from the definitions of sine and cosine and the Pythagorean

Theorem: In right triangle ABC where c is the hypotenuse, by definition $\sin A = \frac{a}{c}$ and $\cos A = \frac{b}{c}$ ; therefore $\sin^2 A + \cos^2 A = \frac{a^2}{c^2} + \frac{b^2}{c^2}$ . Then if $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$, multiplying through by $c^2$ yields $a^2 + b^2 = c^2$, which we know is true because it is simply the Pythagorean Theorem.

### Domain, Range, and Signs of Functions

It's much easier to remember which trig functions are positive and negative in which quadrants if we simply note that in the first quadrant they are all positive; in the second quadrant only sin and its reciprocal, csc, are positive; in the third quadrant only tan and its reciprocal are positive; and in the fourth quadrant only cos and its reciprocal are positive. This can be summarized by the mnemonic "All Students Take Calculus": All positive, Sin positive, Tan positive, Cos positive.

(The second-best way to figure out which functions are positive where is to think about which functions depend on the $x-$coordinate, which ones depend on the $y-$coordinate, and which ones depend on both, and then figure out which coordinates are negative depending on which quadrant we are considering. For example, in the third quadrant, where both coordinates are negative, sin and cos will be negative (and therefore so will sec and csc), but tan (and therefore cot also) will be positive because $\frac{x}{y}$ is positive when both x and y are negative. But this method takes longer, and is more prone to error, than simply using the mnemonic above.)

### Points to Consider

It can often be easier to tell if an equation is *not* an identity: simply plug in a couple of values for the variables in the equation and see if the equation holds true. If it doesn't, the equation is not an identity; if it does, the equation might or might not be.

Similarly, you can verify the domain and range of a function by plugging in numbers that are inside or outside the domain or range and seeing what happens. If a number is in the domain of the function, it should yield a sensible result when you substitute it for $x$; if it's not in the domain, it should not yield any real-number result. If a number is in the range of the function, you should be able to get it as a result when you plug in some value of $x$; if it's not in the range, you shouldn't be able to for any value of $x$.

# Applications of Right Triangle Trigonometry

### Solving Right Triangles

If you are going through example 2 as a class, you may want to have students stop and think about how they might solve the triangle before walking them through the solution presented in the text. First, they should assess how much information they already have available—in this case, one side and two angles. (They may only notice the one angle measure that is written out in numerals—remind them that they also know the measure of the right angle.) Then they should think about what strategies they know for finding the other sides and angles (Pythagorean theorem, trig ratios), and which numbers they would need to plug in for each of those strategies. Finally, considering what numbers they actually have available to plug in should give them some idea of which strategies they could effectively use.

To answer the question posed in the solution: Using the tangent to find the third side is better than using the sine because using the tangent allows us to plug in the side we were given at the beginning, whose length we know precisely, rather than the side we just found, whose length we only know approximately. It's always best to base our calculations on the most precise information we have available, so that rounding errors don't accumulate.

### Angles of Elevation and Depression

Clinometers are tools for measuring angles of elevation and depression; theodolites can measure horizontal angles as well. More information about both of these can be found on the Internet or in an encyclopedia.

For extra precision when measuring angles of elevation and depression, you should of course subtract several inches

from your total height to estimate the distance from your eyes to the ground.

**Other Applications of Right Triangles**

An explanation of how we know the information given in example 7: We know the distance between the moon and the earth based on calculations that will be explained later in the book. We know the angle between the moon and the sun at a given time because we can measure it directly from our vantage point on the earth. And finally, we know that the moon makes a right angle with the earth and sun at the first quarter (when the moon is halfway full) because exactly half of the portion of the moon that we can see is lit up by the sun, meaning that the sun must be shining exactly "sideways" on the moon.

**Points to Consider**

In addition to the situations described in this lesson, we also might use right triangles to determine the shortest distance between two points on a grid (like a grid of city blocks), or to determine how long a ladder we need to reach a certain height on a building.

Any right triangle can be solved if we have enough information; at minimum, we need to know the length of at least two sides, or one side and one angle besides the right angle.

Trigonometry can solve problems at any scale because the trig ratios are the same for any size triangle as long as the angles are the same.

# 1.2  Circular Functions

## Radian Measure

**Radians, Degrees, and a Calculator**

Most scientific and graphing calculators have a $\pi$ key, primarily to make calculating angles in radians easier. Make sure your students know where this key is on their calculators.

When a question like example 4 comes up, students may wonder how they are supposed to know that this angle measure is in radians and not degrees. After all, $\frac{3\pi}{4}$ is just a number like any other, so an angle of $\frac{3\pi}{4}$ degrees could exist too—which means we can't just assume any angle measure with $\pi$ in it is in radians. And as we've seen in the text, measures like 1 radian and 2 radians are meaningful as well, so we can't assume any angle measure *without* $\pi$ in it is in degrees.

The solution to this conundrum is to assume all angle measures are in radians unless otherwise specified; that's the convention used by mathematicians. So if you're asked what the cosine of 5 is, if the problem says just [U+0080][U+009C]5[U+0080][U+009D] and not [U+0080][U+009C]5°," assume it means 5 radians. But beware of typographical errors! If an angle measure doesn't include a degree sign, but is a suspiciously familiar round number like 60 or 90, it may be worth looking over the problem for signs that the author might have really meant to specify degrees and just left out the degree sign by mistake.

Example 7 contains a new concept: the inverse sine function. You may need to help students find the inverse sine on their calculators (usually they'll need to press the [U+0080][U+009C]$2^{nd}$[U+0080][U+009D] key followed by the "sin" key), and you may also need to explain the inverse sine function itself.

First, a brief review of inverse functions may be needed: remind students that the inverse of a function is simply what you get when you apply the function "backwards," so the input becomes the output and the output becomes the input. In the case of the sine function, normally the input is an angle measure (which can be any real number) and the output is a ratio of side lengths (which can be any real number between 1 and 1). The inverse sine function, therefore, takes a number between 1 and 1 as its input, and its output is the measure of an angle whose sine is that number. For example, the inverse sine of 1 is $90°$ (or $\frac{\pi}{2}$ radians), because the sine of $90°$ (or $\frac{\Pi}{2}$ radians) is 1.

The notation used here also bears explaining. Inverse functions are written with what looks like an exponent: the inverse of $f(x)$ is written as $f^{-1}(x)$, and the inverse of $\sin(x)$ is written as $\sin^{-1}(x)$. Emphasize that $\sin^{-1}(x)$ does not mean $\sin(x)$ raised to the power of $-1$, even though it looks as if it does. (Normally, as we learned in the previous chapter, placing the exponent right after the $f$ in $f(x)$ is the standard notation for raising a function to a power, and when raising a trig function like $\sin(x)$ to a power, we put the exponent right after the "sin" part. But when we want to raise $\sin(x)$ to the power of $-1$, we must write $(\sin x)^{-1}$ instead, so that we can use $\sin^{-1}(x)$ to designate the inverse sine of $x$.)

## Applications of Radian Measure

**Rotations**

Example 1 contains a slight error: since the hour hand has rotated $\frac{1}{3}$ of the way from 11 to 12, the distance between the hour hand and the 12 is $\frac{2}{3}$, not $\frac{1}{3}$, of that twelfth of the circle; it is $\frac{\pi}{9}$ rather than $\frac{\pi}{18}$ radians.

**Length of Arc**

The illustration for example 2 shows 12 feet as the diameter of the circle, but the solution worked out in the text is in fact based on the *radius* of the circle being 12 feet. If you wish, you can have students solve the problem both ways just for practice, but make sure to keep everyone on the same page about which formulation of the problem you are using at which time.

Perceptive students may try to solve example 3 the short way, by noticing that since the radius of the larger circle is $\frac{7}{4}$ that of the smaller circle, the circumference is also $\frac{7}{4}$ as great, and so a complete rotation of the smaller circle would be $\frac{4}{7}$ of a rotation of the larger circle. This is certainly a legitimate way of solving the problem, but you might encourage them to do it over again the way the book describes, just so they can get some practice doing calculations with arc lengths.

Additionally, there is a more precise way to express the answer to this problem. Instead of finding a decimal approximation for θ and then multiplying it by $\frac{180}{\pi}$ to approximate the angle measure in degrees, it is better to simply leave the value of θ in fraction form as $\frac{8\pi}{7}$, so that multiplying it by $\frac{180}{\pi}$ yields the answer $\frac{1440}{7}$ degrees.

**Area of a Sector**

You may need to explain the setup of the equation for 1 radian. Basically, we are starting with the equation $2\pi$ (radians) $= \pi r^2$ (area) and dividing by $2\pi$ to make the left side equal 1 radian. Dividing the right side by $2\pi$ then gives us $\frac{1}{2}r^2$ as the area.

The diagram given for example 4 contains a slight error ; you might encourage students to find it for themselves. (It's a tricky error to catch—just remember that $\frac{2\pi}{3}$ is not in fact $\frac{2}{3}$ of $2\pi$.) This is a good opportunity to remind them not to make the same error themselves, as it's quite a common one.

**Length of a Chord**

Example 5 presents a slightly convoluted solving method, as it has students find *half* the length of the chord and then double it to get the final answer. In future, students will find it easier to simply use the formula for the *whole* chord length: twice the radius of the circle times the sine of half the angle, or $2r \sin\left(\frac{\theta}{2}\right)$.

Also, when applying the chord-length formula, it isn't actually necessary for the angle measure to be in radians (as it is with the arc-length and area formulas), because the sine of the angle is the same whether the angle is in radians or degrees. However, it is still useful to have students practice converting from degrees to radians.

## Circular Functions of Real Numbers

$y = \sin(x)$, **the Sine Graph**

Students should notice that the height of the point tracing out the sine graph at any given stage is exactly the same (in graph-units) as the height of the point moving around the circle. (This may not be immediately obvious because the sine graph and the circle graph are depicted on slightly different scales.)

$y = \cos(x)$, **the Cosine Graph**

The relationship between graph height and location on the circle is harder to see for the cosine graph, because the height of the graph represents the horizontal rather than the vertical location of the point on the circle. Imagining the circle rotated a quarter-turn to the left may help make the connection more visible, as you can now see the heights of the two points matching the same way they did on the sine graph—but your students probably needn't try this, as long as they understand the basic principle that the cosine represents the $x-$value of the corresponding angle.

$y = \tan(x)$, **the Tangent Graph**

It may seem strange that the tangent line can get infinitely long when the sine and cosine lines can't. Remind students that the length of the tangent line represents the ratio between the sine and cosine, and so it gets infinitely big as the

cosine gets infinitesimally small.

To explain this in terms of the similar triangles shown in the diagram: The sides marked $t$ and 1 have the same ratio as the sides marked $y$ and $x$. As side $y$ gets longer, side $t$ gets longer—but as side $x$ gets shorter, side 1 can't get shorter because its length is fixed at 1. So if that side can't get shorter, side t has to get even longer to keep the ratio the same.

**The Three Reciprocal Functions:** $\cot(x), csc(x),$ **and** $\sec(x)$

You might stop and ask your students why it makes sense that 1 and $-1$ are the only values for which a function and its reciprocal are the same. (Hint: What has to be true of $y$ in order for $y = \frac{1}{y}$ to be true? Further hint: What happens when you solve that equation for $y$?)

The illustrations showing the cosecant segment for angles greater than 180° may be a little confusing, as the segment looks the same as it did for angles less than 180°, but its length is now being described as a negative number. The reason it is now negative is that the segment is now pointing in the opposite direction from the line segment that forms the terminal side of the angle, whereas it was pointing in the same direction for angles less than 180°.

To make it extra clear that the graph of the secant function is not made up of parabolas, you can point out that the secant graph has vertical asymptotes, whereas parabolas have no domain restrictions and extend infinitely far in both the positive and negative $x-$directions.

**Lesson Summary**

It's possible to express the domain restrictions on the cotangent and cosecant functions in a way that makes clearer their relationship to the domains of the tangent and secant functions. Instead of $x : x \neq n\pi$, where n is any integer, we can express the domain of the cotangent or cosecant as $x : x \neq n\left(\frac{\pi}{2}\right)$, where $n$ is any even integer. When we compare this to the domain of the secant or tangent, $x : x \neq n\left(\frac{\pi}{2}\right)$, where $n$ is any odd integer, we can see much more clearly that the cosecant and cotangent have the same pattern of asymptotes as the secant and tangent, just shifted by $\pi$ units.

# Linear and Angular Velocity

**Linear Velocity** $v = \frac{s}{t}$

You may need to stress that s represents distance and does not stand for "speed." Students will still be prone to forget this when plugging in values without thinking too hard, so they may need reminding when they slip up.

In example 2, students might need to be reminded that the bar over the 3 in $21.\overline{3}$ signifies a repeating decimal; the quantity being expressed is $21.233333333\ldots$ with the 3's continuing on forever.

**Angular Velocity** $\omega = \frac{\theta}{t}$

After finding the angular velocity in example 4, you may want to pause and explain why this answer makes sense: if the girls rotate through $\frac{1}{6}$ of the circle each second, it will take them 6 seconds to make a complete rotation, and 6 seconds is indeed the time we were given at the beginning of the problem.

Similarly, it is useful to check the linear velocities and see why they make sense. Since Lindsey is 7 feet from the center while Megan is 2.5 feet from the center, Lindsey should be traveling about $\frac{7}{2.5}$ or $\frac{14}{5}$ times as fast as Megan. Sure enough, their speeds are about 7.3 and 2.6 feet per second respectively, and $\frac{7.3}{2.6}$ is about the same as $\frac{14}{5}$.

Another problem for students to consider: if your arms are 2 feet long, and you swing a baseball bat that is also 2 feet long, how much faster is the end of the bat traveling than the end of your arms? (You can make up an angular velocity for the swing and have students find the two linear velocities, but you don't need to—it may be more educational to have them look at the problem more abstractly. The key insight here is that no matter what the actual speeds are, since the arc of the circle that the bat sweeps out has a radius twice as great as the arc of the circle swept out by the swinger's arms, the end of the bat will be traveling twice as fast. Students can see this by looking at the formula for linear velocity in terms of angular velocity, $v = r\omega$, and considering what happens if $r$ is doubled—or multiplied by

any other constant.) What does this suggest about why we use baseball bats—or for that matter, tennis rackets, golf clubs, or croquet mallets?

You may also want to have students try solving the formulas backwards: for instance, if the girls' angular velocity in Example 4 were $\frac{\pi}{2}$ radians per second, what would be their linear velocity if the merry$-$go-round were still the same size?

## Graphing Sine and Cosine Functions

### Amplitude

You may need to stress that the amplitude is the greatest distance the wave gets from the *center* of the wave, so it is only half the distance between the minimum and maximum values.

### Period and Frequency

You may want to clarify the exact relationship between period and frequency, perhaps by helping students to work it out on their own. Remind them that an inverse relationship means that one quantity decreases when the other increases, and that it also means that the two quantities will yield a constant result when multiplied together. The three examples given in the text are a function with a period of $2\pi$ and a frequency of 1; a function with a period of $\frac{\pi}{4}$ and a frequency of 8; and a function with a period of $4\pi$ and a frequency of $\frac{1}{2}$. In each of these cases, what do we get when we multiply the period by the frequency? (Answer: $2\pi$.) So what does that suggest the relationship is between the two? (Answer: $\text{period} * \text{frequency} = 2\pi$, which is more usefully expressed as either $\text{period} = \frac{2\pi}{\text{frequency}}$ or $\text{frequency} = \frac{2\pi}{\text{period}}$; both expressions are useful in different situations.)

$$y = csc(x)$$

Students may find it a little counterintuitive that the period of the cosecant graph is $2\pi$, because the graph divides up so neatly into $\pi-$sized chunks (and also perhaps because they've just seen that the period of the tangent graph is $\pi$ units). Stress that it takes two of those chunks, in this case, before the graph actually repeats itself, just as in Example 2 at the beginning of the lesson it took one "high" portion and one "low" portion of the graph together to make up one period.

### Transformations of Sine and Cosine Graphs: Dilations

It's a little hard to tell from the graphs of those linear functions that the line is being "stretched" vertically when the slope increases, because vertically stretching a straight line looks just like rotating it. You may want to also show graphs of the dilations of $x^2$ discussed in the text, because those graphs will make the stretching and shrinking a lot clearer.

Also, the statement "Constants greater than 1 will stretch the graph out vertically and those less than 1 will shrink it vertically" is really only part of the story. That is, it's true for positive constants, but negative constants will first flip the graph upside down and then stretch or shrink it vertically (stretch it if their absolute value is greater than 1, shrink it if it's less than 1.) And of course, a constant of 0 will shrink the graph all the way down to a straight horizontal line.

### Review Questions

Encourage students to sketch graphs for question 2. Also, remind them if necessary that an amplitude of $A$ units (where $A$ is the constant multiplier in front of the trig function) means that the graph goes both $A$ units above the $x-$axis and $A$ units below it, so the maximum is $A$ and the minimum is $-A$. (The exception, of course, is when $A$ is negative, as in part c; then $A$ is the minimum and $-A$ is the maximum.)

## Translating Sine and Cosine Functions

**Vertical Translations**

Another way to find the answer to example 1 is to find the minimum and maximum of the base function, $\cos(x)$, and then subtract 6 from both of them.

**Horizontal Translations (Phase Shift)**

Here is another way to explain the apparent "backwards-ness" of phase shift: when we look at a given number $x$, an expression like $\cos(x-2)$ means "the cosine of the number that's 2 units to the left of this one." So it's as though we're taking the value of the cosine function from 2 units to the left of "here," and moving it over to "here"—which means we're moving it 2 units to the right of where it started out. And of course we're doing the same thing with the whole function, so $\cos(x-2)$ describes the whole cosine function shifted 2 units to the right.

Yet another way to explain it is in terms of inverse functions. Students may remember from previous algebra classes that the graph of an inverse function is the graph of the original function reflected about the line $y = x$; in other words, flipped diagonally. This means that a horizontal shift of the original graph would result in a vertical shift of the inverse graph, and vice versa. Specifically, shifting the original graph to the right corresponds to shifting the inverse graph up, and shifting the original graph left is the same as shifting the inverse graph down.

Now, one way to find the equation for an inverse function is to swap the $x-$ and $y-$values of the original equation and then solve for $y$ — so if the original equation was $y = 3x$, the inverse equation would be $x = 3y$, or in other words $y = \frac{x}{3}$. But note that if we add a constant to the $x-$value in the original equation, that constant ends up being subtracted from the inverse equation—if we start with $y = 3(x+4)$ instead of just $y = 3x$, the inverse is $x = 3(y+4)$, or $x = 3y+12$, or $x-12 = 3y$, or $y = \frac{x}{3}-4$. Because everything gets reversed in an inverse operation, *increasing* the $x-$value of the original function means decreasing the $y-$value of the inverse function. That means the inverse function gets shifted down (not up), and that must mean the original function was shifted left (not right) when the $x-$value was increased. So, increasing the $x-$value means shifting the graph left, and vice versa.

One way to explain why sin and cos are just phase-shifted versions of each other is to recall that they are based on the $x-$ and $y-$coordinates of a point moving around the unit circle. These coordinates behave in the same way as the point rotates—they both oscillate between 1 and $-1$—it's just that they start out at different points in the cycle.

Another way to explain it is this: Recall that the sine of an angle is the cosine of the angle's complement (draw a triangle to see why this is so—the sine of one acute angle is the cosine of the other acute angle, and the two angles add up to $90°$ by the Triangle Sum Theorem). We can write this fact as $\sin(x) = \cos(90-x)$, and we can rewrite $\cos(90-x)$ as $\cos(-x+90)$. But the graph of $\cos(-x+90)$ is simply the graph of $\cos x$ flipped over vertically (since all the negative $x-$values become positive and vice versa) and shifted horizontally by $90°$ (or $\frac{\pi}{2}$ radians), and that happens to be the same as the graph we get if we shift it $\frac{\pi}{2}$ units in the other direction and *don't* flip it.

## General Sinusoidal Graphs

**Drawing Sketches/Identifying Transformations from the Equation**

Some students may notice that translating a basic sine or cosine graph $\pi$ units horizontally is essentially the same as flipping it upside down (i.e. multiplying it by $-1$). This isn't a necessary fact to know, but it can be useful. For particularly curious students, here's how to explain why it is true:

When you add $\pi$ to an angle measure, you get another angle two quadrants away with the same reference angle. (Sketching a couple of angles will make this obvious.) That means all the trig functions for that new angle will be

either the same or the opposite of the trig functions for the old angle. Now, in any two quadrants that are opposite each other, the sine function has opposite signs—it's positive in I but negative in III, and positive in II but negative in IV. Similarly, the cosine function is positive in I but negative in III, and negative in II but positive in IV. So, whenever we add $\pi$ units to an angle, the sine and cosine of the new angle are the negatives of the sine and cosine of the old angle—or, to put it more formally, for any angle $x$, $\sin(x+\pi)$ equals $\sin(x)$ and $\cos(x+\pi)$ equals $-\cos(x)$. In other words, applying a phase shift of $\pi$ units to the sine or cosine graph is the same as multiplying it by $-1$, or flipping the graph over.

(You may also notice that, since the sign of the tangent function alternates from quadrant to quadrant, the tangent function keeps the same sign when you add $\pi$ to the angle. This is why the tangent function simply repeats itself after $\pi$ units, and a phase shift of $\pi$ units is equivalent to no change at all.)

Examples 1 and 2 demonstrate two different ways to approach the problem of sketching a graph: starting with the horizontal and vertical translations, or starting with the amplitude and frequency. Students will probably find they prefer one method or the other, and there's certainly no need to be strict about which one to use.

Also, some students may find it easier to sketch a complete curve at each step of the process until they end up with the final curve, while others may find it easier to simply sketch the key points of the graph, move them around as necessary, and not connect them with a curve until the final step. Again, either method should work fine; it may in fact be a good idea to point out that both methods exist.

### Writing the Equation from a Sketch

A second way to find the amplitude after finding the phase shift is just to subtract the phase shift from the maximum value of the function. For instance, in the example given, we would subtract 20 from 60 to get 40.

### Review Questions

Problems 6-10 should contain the instruction "Write an equation that describes the given graph." Of course, either sine or cosine may be used.

# 1.3 Trigonometric Identities

## Fundamental Identities

**Reciprocal, Quotient, Pythagorean**

This section reviews the definitions of the trig functions and the Trigonometric Pythagorean Theorem. The Pythagorean identities, you'll recall, were first covered in lesson 1.6, but here we see a slightly different way of deriving them. It may be useful to reinforce knowledge of the identities by walking through the derivations, but for many students this will simply be review.

You can point out that another way of expressing the reciprocal trig functions is as follows: $\csc\theta = \frac{c}{b}; \sec\theta = \frac{c}{a}; \cot\theta = \frac{a}{b}$. This way of expressing them is only useful for angles in triangles, though; for angles of rotation it may be more useful to think of them as $\frac{1}{\sin\theta}$ and so on.

**Confirm Using Analytic Arguments**

The diagram here with the vertical line representing a distance of t units may confuse students a little, but it simply demonstrates in a slightly unusual way the fact that any real number can correspond to a distance traveled around the unit circle, and therefore to an angle on that circle. Again, this should be review for most students.

There is a slight error in the last paragraph : where it reads "for points in the third and fourth quadrants we use angles formed by the radius that meets that point and the y axis," it should say "x axis." This is an opportunity to remind students about reference angles: the reference angle is always the angle made with the closest portion of the $x-$axis.

**Confirm Using Technological Tools**

Calculators generally do not have sec, csc, and cot keys, so instead one must use the cos, sin, and tan keys and then take the reciprocals of those functions using the reciprocal key (marked $x^{-1}$ or $\frac{1}{x}$). Stress once again that the keys marked $\sin^{-1}, \cos^{-1}$, and $\tan^{-1}$ will not yield the reciprocal functions $\csc, \sec$, and tan, but rather the arcsin, arccos, and arctan (that is, inverse sine, cosine, and tangent) functions.

**Alternative Forms**

We see here that knowing just one trig function of an angle does not uniquely determine the angle, as there are always two quadrants it could be in (which two depends on whether the value of the trig function is positive or negative). Generally, knowing a second trig function of the angle, or at least knowing its sign, will narrow down which quadrant the angle could be in—but note that this won't be the case if the second function is just the reciprocal of the first function. So, for example, knowing the signs of the tangent and cotangent functions won't tell us what quadrant the angle is in, because the tangent and cotangent always have the same sign whatever the quadrant—and the same is true for the sine and cosecant, or the cosine and secant. But knowing the signs of, say, the tangent and secant functions will tell us which quadrant the angle is in, and the same is true for any two trig functions that are not each other's reciprocals.

## Verifying Identities

**Working with Trigonometric Identities**

Students may get a little confused by the fact that the equations we're working with reduce to seemingly obvious identities. The point here, of course, is that we start with an equation that declares two complex expressions to be equivalent, and then prove the equation is true by showing it can be changed into a form that is much more obviously true.

Generally, as the text states, we only change one side of the equation at once. This is because we are not really changing the equation, merely changing how the equation is expressed. To take a simpler example: if we were asked to prove that $17 [U+0080] [U+0093] \left(\frac{99}{3}\right) + 2(40 + 7) = 78$, we would not try to perform the same operations on both sides at once, because we don't need to; one side is already as simple as it can get. Instead, we would simplify the left-hand side, step by step, until we had shown that it does indeed equal 78, and so is the same as the right-hand side.

Here's one way to explain the difference between proving identities and solving equations: When we start out with a complicated equation like $17x + 5(x - 2) + 3 = 15$ and solve it for $x$, what we're really trying to do is answer the question "For what value(s) of $x$ is this equation true?" When we reduce the equation to $x = 1$, we've just shown that the original equation is true if $x = 1$, and false otherwise.

But if we tried to solve an equation like $3(x + 5) = 5x [U+0080] [U+0093] 2(x + 5) + 25$, we would find that it reduces to $3x + 15 = 3x + 15$, which reduces even further to $0 = 0$, which is true no matter what $x$ is. So instead of proving the equation is true for certain values of $x$, we've discovered it's true for *all* values of $x$.

And that's exactly what we're doing in proving these trig identities: by showing that both sides of the equation reduce to the same expression, we're proving that the equation is always true for any value of $\theta$. And once we know it's true, we can use it to make useful substitutions when solving trig problems in the future.

**Technology Note**

Calculators can be useful in verifying identities, but it is dangerous to rely on them too much. If the graphs of two expressions look identical, it may mean the expressions are indeed equivalent, but it may also mean that the difference between them is just too small for the graph to show, or that they are only equivalent over this small interval. Since a graphing calculator can only show us part of a graph and can only draw it with limited precision, it cannot tell us for sure if two expressions really are mathematically equivalent.

What a graph can do, though, is tell us for sure if two expressions are *not* equivalent. If the graphs of the expressions look wildly different—in fact, if they look even a little bit different—then we can safely say that the expressions are not equal.

## Sum and Difference Identities for Cosine

**Difference and Sum Formulas for Cosine**

Students may not immediately see why we are trying to find a formula for $\cos(a - b)$. The idea is that once we find such a formula, we can use it to find the cosine of an unfamiliar angle if we can express that angle as the difference of two angles we are familiar with. The sum formula will be similarly helpful.

Students who haven't encountered the distance formula for a while might need to be reminded of it. On the coordinate grid, the distance between two points $(a, b)$ and $(c, d)$ is equal to $\sqrt{(a - c)^2 + (b - d)^2}$; in other words, the square of the straight-line distance between the points is equal to the square of the horizontal distance plus the square of the vertical distance. This is derived directly from the Pythagorean Theorem; drawing a right triangle on the coordinate grid whose hypotenuse is the line between the two points will show how it is derived.

Note that two different identities are derived in this section. The diagrams and the table show how we derive the formula for $\cos(a - b)$; then the following lines show how we can use this formula to derive, in turn, the formula for $\cos(a + b)$. These identities may be easier for students to remember if expressed in words: "The cosine of the difference is the product of the cosines plus the product of the sines" and "The cosine of the sum is the product of

the cosines minus the product of the sines."

**Use Cosine of Sum or Difference Identities to Verify Other Identities**

Call attention to the labels "Identity A" and "Identity B" here. These labels aren't official names for these identities, but they will be used to refer back to them later in this lesson.

The identities themselves simply say that the sine of an angle is equal to the cosine of the angle's complement, and vice versa. We've already seen that this is true from working with angles in right triangles, as the sine of one acute angle in a triangle is equal to the cosine of the other, and the two angles are each other's complements.

**Use Cosine of Sum or Difference Identities to Find Exact Values**

Here we see that the sum and difference formulas we have just learned can tell us the value of the cosine function for angles we haven't previously worked with, based on the values we already know for angles that are multiples of $30°$ or $45°$. (Note, though, that this technique will still only tell us the cosines of angles that are multiples of $15°$. That's because when you add or subtract angles that are multiples of $15°$ (which includes all angles that are multiples of $30°$ or $45°$), the result is always also a multiple of $15°$.)

**Technology Note**

As before, note that calculators cannot tell us with absolute certainty if two expressions have identical values; they might, for example, be identical up to twenty decimal places, but differ after that point. However, if two expressions seem to have equal values when plugged into a calculator, there is a reasonably good chance that they are really equal—and more importantly, the calculator will clearly tell us if they are not equal at all. Hence, double-checking answers with a calculator is still a good idea.

# Sum and Difference Identities for Sine and Tangent

**Sum and Difference Identities for Sine**

(For reference, Identities A and B are on page 244.)

The angle in example 1, $\frac{5\pi}{12}$, could of course be expressed in many different ways as the sum or difference of two other angles, but the way shown in the text is the most convenient way to express it in terms of angles we are already familiar with. Consider asking your students to explain why this choice is a good one. (You might need to remind them, though, that $\frac{3\pi}{12}$ is the same as $\frac{\pi}{4}$ and $\frac{2\pi}{12}$ is the same as $\frac{\pi}{6}$ in order for them to see why these angles should be easy to work with.)

A shortcut to finding the cosines of the angles in example 2 is to notice that the side lengths are Pythagorean triples. That is, if the sine of angle $\alpha$ is $\frac{5}{13}$, the other leg of the relevant triangle must measure 12 (to complete the triple $5 - 12 - 13$) and so the cosine is $\frac{12}{13}$. Similar reasoning holds for angle $\beta$.

**Sum and Difference Identities for Tangent**

Once again, expressing the sum formula for tangent in words may make it easier for students to remember: "The tangent of the sum equals the sum of the tangents over 1 minus the product of the tangents." There isn't really a concise way to do this for the sine formulas, though.

Consider whether you want your students to memorize these and other trig identities or not. They definitely need to develop a good sense for when to use which ones, but the formulas themselves are the least important part of that knowledge; in fact, knowing how to derive the formulas may be more useful than simply knowing the formulas themselves. Instead of requiring that all the formulas be memorized for a test, for example, it might be more educational to supply a few of the formulas and require that students re-derive the others from the few given (after making sure, of course, that the formulas supplied are sufficient to derive the others from.)

Of course, some students might find that a little too challenging, but on the other hand, some will find it easier than

memorization. Perhaps a good compromise would be to provide a few of the formulas so that students can derive the others if they want to, but not make that derivation a required part of the exam. Since students will presumably need to use any or all of the formulas to solve some of the exam questions, they will still have to either derive the formulas or have them memorized, but can do whichever of those two works best for them.

## Double-Angle Identities

### Deriving the Double-Angle Identities

There is really only one form of the double-angle identity for sine, but we see here that the double-angle identity for cosine can be expressed in several different ways. Students should be familiar with all of these formulas in case they encounter them "in the wild"; it's good to be able to recognize when an expression can be converted to $\cos 2a$, just as it's useful to be able to express $\cos 2a$ in terms of functions of a. Also, as will be seen later, some problems are easier or harder depending on which form of the double-angle identity we choose to work with.

### Applying the Double-Angle Identities

Just as in the previous lesson, noticing that 5 and 13 are part of a Pythagorean triple is a useful shortcut to finding $\cos a$.

### Finding Angle Values Given Double Angles

(Example 1 should read "$2x$ is a Quadrant II angle" (rather than "$x$ is a Quadrant II angle"), both in the introduction and in the second-to-last line of the table.)

The seventh line of the table may merit an explanation, as students whose algebra is rusty may wonder where the [U+0080][U+009C]$a$[U+0080][U+009D] comes from. Explain that it is simply a "dummy variable" that we choose to stand for $\sin^2 x$ so that the equation we are working with becomes a simple quadratic, which we know how to solve. Later, we will change $a$ back to $\sin^2 x$ in order to finish solving for $x$.

### Simplify Expressions Using Double-Angle Identities

In the example given here, you can show, if you wish, how choosing a different double-angle formula would make the expression we are working with more complicated, and hence why the one used here is the best one to use in this case.

For example, using $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ instead yields the following:

$$
\begin{aligned}
&\frac{1 - \cos 2\theta}{\sin 2\theta} \\
&= \frac{1 - (\cos^2 \theta - \sin^2 \theta)}{2 \sin \theta \cos \theta} \\
&= \frac{1 - \cos^2 \theta + \sin^2 \theta}{2 \sin \theta \cos \theta} \\
&= \frac{1}{2 \sin \theta \cos \theta} - \frac{\cos^2 \theta}{2 \sin \theta \cos \theta} + \frac{\sin^2 \theta}{2 \sin \theta \cos \theta} \\
&= \frac{1}{2 \sin \theta \cos \theta} - \frac{\cos \theta}{2 \sin \theta} + \frac{\sin \theta}{2 \cos \theta} \\
&= 2 \csc \theta \sec \theta - \frac{1}{2} \cot \theta + \frac{1}{2} \tan \theta
\end{aligned}
$$

and at this point it isn't at all obvious how to show that this is equivalent to $\tan \theta$.

### Lesson Summary

The first paragraph here describes a useful strategy for getting an idea of what sort of answers are reasonable before attempting to solve a trig problem: figure out, if you can, approximately where the given angles are and approximately what the values of the trig functions should be for those angles, or at least figure out upper or lower bounds on the trig functions based on whether the given angle is greater or less than an angle you are already familiar with, and whether the relevant trig functions are increasing or decreasing in the part of the unit circle where the angle is located.

This particular example contains a slight error , though: $\sin 45°$ is actually about 0.7, and $\sin 30°$ is 0.5, so the angle $\theta$ in the problem is actually between $30°$ and $45°$, and $2\theta$ is therefore between $60°$ and $90°$.

**Review Questions**

Some of these questions involve finding $\tan 2x$, which we haven't derived a formula for. In fact, there isn't a concise formula for the tangent of a double angle; however, once $\sin 2x$ and $\cos 2x$ have been found, $\tan 2x$ is simply the quotient of the two.

## Half-Angle Identities

**Deriving the Half-Angle Formulas**

In the first and second derivations here, it's important for students to see that when we substitute a for $2\theta$, we can also substitute for $\theta$. This is how we are able to find an expression for the sine or cosine of $\frac{a}{2}$ based on that of $a$.

(In the tangent derivation, though, the text uses $a$ and $\theta$ interchangeably. This is because we don't have to make any actual half-angle substitutions there, but merely plug in the formulas for sine and cosine.)

Expressing the half-angle formula for cosine as $\sqrt{\dfrac{1+\cos a}{2}}$ instead of $\sqrt{\dfrac{\cos a+1}{2}}$ may make it easier to remember, as it then looks more like the formula for sine.

**Use Half-Angle Identities to Find Exact Values**

You might need to review how we know the sine of $225°$. $225°$ is equal to $180°+45°$, which means it is a third quadrant angle with reference angle $45°$, but you may want to draw it to make this clearer.

**Find Half-Angle Values Given Angles**

For example 2, stress that although we have enough information to figure out the measure of angle $\theta$, we don't actually need to know it to apply the half-angle formula; the formula only requires us to know the cosine of $\theta$, which we already have.

You may want to explain, though, how we know that half of a fourth quadrant angle is a second quadrant angle. In general, halving a first or second quadrant angle will yield an angle in the first quadrant, and halving a third or fourth quadrant angle will yield an angle in the second quadrant. We can verify this numerically: half of an angle between $0°$ and $180°$ must be between $0°$ and $90°$, and half of an angle between $180°$ and $360°$ must be between $90°$ and $180°$.

**Using the Half- or Double-Angle Formulas to Verify Identities**

Again, you may point out in the second line of this derivation that of the three possible expressions for $\cos 2\theta$, we've chosen the one that makes the numerator reduce to the simplest form.

**Technology Notes**

Another way to demonstrate that $sin\frac{\theta}{2}$ does not equal $\frac{1}{2}\sin\theta$ is to pick a familiar pair of angles, such as $90°$ and $180°$, and note that $\sin 90°$ is definitely not half of $\sin 180°$.

*1.3. TRIGONOMETRIC IDENTITIES*

# Product-and-Sum, Sum-and-Product and Linear Combinations of Identities

**Transformations of Sums, Differences of Sines and Cosines, and Products of Sines and Cosines**

Remind students not to mix up the formulas here with the sum and difference identities learned earlier: the formula for $\cos(\alpha + \beta)$ is not at all the same as the formula for $\cos\alpha + \cos\beta$!

For an extra challenge, you might ask students to derive the three formulas whose derivations are not shown, by applying similar reasoning to that used in the derivation that is shown.

**Transformations of Products of Sines and Cosines into Sums and Differences of Sines and Cosines**

The product formulas shown here bear a certain resemblance to the sum formulas they are derived from, and students may be tempted to apply them after applying the sum formulas. For example, they may think, after they have determined that $\cos(a + b) = \cos a \cos b - \sin a \sin b$, that it would then be a good idea to plug in the expressions they've just learned for $\cos a \cos b$ and $\sin a \sin b$. But this will only give them a complex expression in terms of $\cos(a + b)$ and $\cos(a - b)$, which won't help them much since the value of $\cos(a + b)$ is what they were looking for to begin with.

The key thing they need to understand is that each of these identities is a tool to be used in different situations, depending on what knowledge they already have. If they need to find the sine or cosine of an angle, and that angle can be expressed as a sum or difference of two angles of which they already know the sine and cosine, then the sum formula from earlier is useful. If they need to know the product of two sines or cosines, and they don't know the sines or cosines themselves, but do know the sine and cosine of the sum and difference of those two angles, then the product formula learned here is useful. In general, they should form the habit of writing down exactly what it is they are looking for and then considering which tools in their possession might apply to that particular situation.

**Linear Combinations**

You may need to define the term "linear combination" for students who haven't encountered it before. A linear combination of two quantities is simply a multiple of one quantity plus a multiple of the other—so, for example, a linear combination of $\sin x$ and $\cos x$ would be any expression of the form $a \sin x + b \cos x$, where $a$ and $b$ are any two numbers (real numbers, unless otherwise specified). Students will use linear combinations in their study of vectors in the next chapter.

# 1.4 Inverse Functions and Trigonometric Equations

## Inverse Trigonometric Functions

**Inverse Functions**

The definition of "one-to-one" bears reviewing here; a function is one-to-one if, in addition to having at most one $y-$value for every $x-$value, it also has at most one $x-$value for every $y-$value.

Students may get confused about the "inverse reflection principle." It doesn't mean that the graph of $f(x)$ or the graph of $f^{-1}(x)$ is symmetric about the line $y = x$, even though in this case both graphs look as though they almost are. Instead, it means that the graph of $f^{-1}(x)$ is what you get when you flip the graph of $f(x)$ about that line. Comparing the two graphs shown in the text should make this clearer.

The inverse reflection principle, in turn, should make it clear why a function has to pass the horizontal line test in order for its inverse to also be a function. When the graph of a relation is flipped about that diagonal line to create the graph of the inverse relation, any horizontal lines drawn through the graph of the original relation would become vertical lines drawn through the graph of the inverse relation—so if the original relation would not pass the horizontal line test, it follows that the inverse relation would not pass the vertical line test, and so would not be a function.

When students work through the examples in the text, they should make sure their calculators are in degree mode. With a calculator, it is much easier to find inverse trig functions in degrees than in radians, because a calculator in radian mode will generally only give decimal approximations for the answers instead of telling you precisely what the answers are in multiples of $\pi$. (When a calculator tells you that the inverse cosine of a number is 1.308996938, it's not easy to guess that that's the same as $\frac{5\pi}{12}$.) Some models of calculator will give answers in multiples of $\pi$, so students who are using those models may not have any difficulty with radian mode, but they should still generally stick with degree mode in order to be on the same page with everyone else.

**Points to Consider**

The inverse relations of the trig functions are one-to-one, but they are not functions. Under the restricted domains that will be discussed later, they are one-to-one functions.

**Using the "Inverse" Notation**

The graphs of $\sin(x)$ and $\sin^{-1}(x)$, as shown, do not look quite like they are each other's reflections about the line $y = x$, but that is simply because they are drawn on different scales. Stretching the graph of $\sin^{-1}(x)$ in the $y-$direction and shrinking it in the $x-$direction would make its resemblance to the graph of $\sin(x)$ more obvious. The same holds for the inverse cosine and tangent graphs.

**Points to Consider**

We can determine exact values for inverse trig functions when those functions correspond with the various special angles on the unit circle.

## Exact Values of Inverse Functions

*Ranges of Inverse Circular Functions*

**Domain and Range of the Circular Functions and their Inverses**

Remind students if necessary that the notation $|q|$ stands for the absolute value of $q$, which is the positive difference between $q$ and zero—in other words, $q$ if $q$ is positive, and $-q$ if $q$ is negative (so $|q|$ is always positive).

*Exact Values of Special Inverse Circular Functions*

**Introduction**

A much easier way to do the problems in example 1 is to think of cot, csc, and sec as being $\frac{adj}{opp}, \frac{hyp}{opp}$, and $\frac{hyp}{adj}$ respectively (we can derive these easily from the definitions of tan, sin, and cos), so that we don't have to find the reciprocals later by dividing fractions.

**Points to Consider**

The inverse composition rule has not previously been discussed in this book, although students may have encountered it in other courses. It states that if the functions $f(x)$ and $g(x)$ are inverses of each other, then $f(g(x))$ and $g(f(x))$ both simply equal $x$. In other words, if we take an $x-$value, apply a function to it, and then apply that function's inverse to the result, we should get the same $x-$value back again. This makes sense since the inverse of a function is really just the function itself applied "backwards."

For example, the inverse of $f(x) = 3x$ is $g(x) = \frac{x}{3}$. According to the inverse composition rule, therefore, both $3\left(\frac{x}{3}\right)$ (that is, $f\left(\frac{x}{3}\right)$) and (that is, $g(3x)$) should equal $x$[U+0080][U+0094] and indeed they do.

So, does the rule apply to trig functions? Yes, if we use the appropriate domain restrictions. For example, the tangent of $45°$ is 1, and the inverse tangent of 1 is $45°$, so $\tan(\tan^{-1}(1)) = \tan(45°) = 1$, and $\tan^{-1}(\tan(45°)) = \tan^1(1) = 45°$. But the rule doesn't apply if we pick an angle measure outside of our restricted domain. The tangent of $225°$ is also 1, but the inverse tangent of 1 is still $45°$. (Of the many angles whose tangent is 1, $45°$ is the only "official" inverse tangent of 1. Similarly, although 2 and $-2$, when squared, each yield 4, 2 is the only "official" square root of 4.) So $\tan^{-1}(\tan(225°))$ would equal $45°$ instead of $225°$.

(This rule will be explored in more detail in the next section.)

## Properties of Inverse Circular Functions

*Derive Properties of Other Five Inverse Circular Functions in terms of Arctan*

**Composing Trigonometric Functions with Arctan**

The notation in this section may be a bit confusing; students may wonder why we would want to find the values of expressions like $\sin(\tan^{-1}(x))$ and $\csc(\tan^{-1}(x))$, or may be unclear on what these expressions really mean. Basically, all we are doing here is the same thing we did in chapter 1, when we were given, say, the sine of an angle and had to find the cosine or the tangent of that same angle without knowing the measure of the angle itself. Based on our knowledge of right triangles and the Pythagorean Theorem, we know that if the tangent of an angle is $x$, we can find all the other trig ratios in terms of $x$ just by drawing an appropriate triangle. If one leg of the triangle measures $x$ units and the other measures 1 unit (making the tangent $\frac{x}{1}$, or just $x$), then we can find the length of the hypotenuse, and once we know the lengths of all three sides, all the trig functions are simply ratios of certain pairs of sides.

So, for example, part b of Example 1 simply asks "What is the sine of an angle whose tangent is 1?" and the answer can be found by drawing a triangle whose legs both measure 1, finding the length of the hypotenuse, and then finding

the sine of the acute angle.

**Points to Consider**

It is indeed possible to graph these composite expressions, as they are simply algebraic functions. Analyzing them will show that their domains are unlimited but their ranges are limited, because the expression $(x^2+1)$ can only take on values greater than or equal to 1.

*Derive Inverse Cofunction Properties*

**Cofunction Identities**

Technically, the graph of $y = \sin\theta$ is not really the graph of $y = \cos\theta$ shifted $\frac{\pi}{2}$ units to the right, but rather the graph of $y = \cos\theta$ flipped upside down and shifted $\frac{\pi}{2}$ units to the left. Since the resulting graph is exactly the same, though, you needn't stress this technicality, but it's useful to keep in mind in case some sharp student spots it. (You might, in that case, challenge them to work out why those two different operations are equivalent.)

*Find Exact Values of Functions of Inverse Functions using Pythagorean Triples.*

The phrase "functions of inverse functions" may be confusing. In this case it simply means "trig functions applied to inverse trig functions," or in other words "inverse trig functions plugged into trig functions"—for example, $\cos(\sin^{-1}(x))$ or $\tan(\cos^{-1}(x))$—which is just the sort of thing we were working with earlier in the lesson.

# Applications of Inverse Circular Functions

*Revisiting* $y = c + a\cos b(x-d)$

**Transformations of** $y = \cos x$

You may want to walk through examples 1 and 2 in more detail, as these concepts haven't been covered since chapter 2.

For example 1, first we need to pull out the amplitude, frequency, phase shift, and vertical shift from the equation: they are $3, 2, \frac{-\pi}{4}$, and 5 respectively. Then we need to think about how all of these values affect the basic cosine graph: it will be stretched vertically by a factor of 3, compressed horizontally by a factor of 2, shifted $\frac{\pi}{2}$ units to the left, and shifted 5 units up. Only after determining all this can we actually draw the graph.

For example 2, we perform almost the same procedure in reverse. First we inspect the graph to discover the amplitude, frequency, phase shift, and vertical shift: they are $5, 4$ (because the period is $90°$), $-20°$, and $-3$, respectively. Then we recall where those numbers fit into the equation, and then we can write the equation. (The equation given in the text should say $20°$ where it says $70°$.)

**Points to Consider**

Given an equation for $y$ in terms of $x$, it is generally possible to solve for $x$ in terms of $y$ (that is, to find the inverse of the equation); the result simply may not be a function. In this case, since we know we can find the inverse of the plain old cosine function, it seems reasonable that we can still find an inverse of the cosine function when it is stretched and shifted as it is here. In the next part of this lesson, we will find out how.

*Solving for Particular Values in Trigonometric Equations*

**Points to Consider**

Degrees and radians are simply two different ways of expressing the same angle measures, so anything that can be done when working in degrees can also be done when working in radians.

*Applications, Technological Tools*

**Examples**

*1.4. INVERSE FUNCTIONS AND TRIGONOMETRIC EQUATIONS*

In Example 1, 4.34 seconds is one time when the dolphin is at a height of 4 feet, but it isn't the first time or even the only time. Try having students also determine the first time the dolphin reaches that height, and two other times when it does so again.

(They may think that since the graph has a period of 3 seconds, the dolphin reaches a height of 4 feet every 3 seconds. This is only partly true; the dolphin does reach that height at 4.34 seconds and every 3 seconds thereafter, but it also reaches that height at 3.66 seconds and every 3 seconds thereafter. Any height on the graph that is not a maximum or minimum will be reached not once, but twice per period: once on the way up and once on the way down. Tracing the graph should confirm this.)

# Trigonometric Equations

*Solving Trigonometric Equations Analytically*

**Introduction**

Before beginning this lesson, you will probably need to review all of the trig identities that were introduced in chapter 3: the Pythagorean identities, sum and difference identities, double- and half-angle identities, and sum and product identities. (If you haven't covered that chapter, you'll need to introduce those identities some other way, as they will be needed to solve the problems in this lesson—except for the double- and half-angle identities, which are covered in the next lesson.)

Students may try applying trigonometric identities to example 2. That isn't a useful technique, though, because the only trigonometric expression in the equation is already as simple as it can get. All that remains is to perform basic algebra to reduce the left-hand side of the equation to $\tan x$, and then to take the inverse tangent of both sides to get $x$ by itself. Except for the inverse tangent step, this problem has more to do with algebra than with trigonometry (proving that algebra skills are essential even as we study more advanced math!).

(Without a calculator, students may also forget the inverse tangent of $\sqrt{3}$. Remind them if necessary to draw an appropriate right triangle to figure it out.)

**Points to Consider**

Any equation-solving method that works for non-trigonometric equations should also work for trigonometric ones, if applied appropriately. And it is certainly possible that a quadratic equation with trigonometric expressions in it might turn up—for example, $\sin^2 x + \sin x + 2 = 0$. We would first solve this equation for $\sin x$, using any standard technique for solving quadratics, and then we would take the arcsine of both sides to find $x$.

*Solve Trig Equations (Factoring)*

**Introduction**

"Principal values," in case students are confused by this, is simply another term for the values that are in the limited domain of a trig function, or in other words are in the range of the inverse function. In example 1, when $\sin x = \frac{1}{2}$, the principal value of $x$ is $\frac{\pi}{6}$ or $30°$, because that is the one $x-$value between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ whose sine is $\frac{1}{2}$.

**Points to Consider**

The quadratic formula will work on trig equations; we just need to remember that we are solving for $\sin x$ rather than for $x$, and still need to take the arcsine afterward.

# Trigonometric Equations with Multiple Angles

*Solve equations (with double angles)*

**Double Angle Identity for the Sine Function**

Example 1, in addition to the double angle identity, also employs algebraic techniques that students may not have encountered for a while, such as the zero product property. It might be necessary to go through the last few steps slowly.

**Double Angle Identity for the Cosine Function**

There is a typo in the first formula derived here: the final line of the derivation should have a minus sign in place of the second equals sign. Also, there should be a line break after the line in the next paragraph that reads $\cos 2a = 2\cos^2 - 1$; the next three lines are a separate derivation of a different formula. (All three formulas are summarized immediately below.)

To solve example 2, we must note that $4\theta$ is twice $2\theta$, so the double-angle formula applies if we plug in $2\theta$ in place of the usual $\theta$.

**Double Angle Identity for the Tangent Function**

You may want to point out that the solution to example 4 makes a great deal of sense if you think of angles as rotations around the unit circle. Remember that $\sin x$ is equal to the $y-$coordinate of the point on the unit circle that corresponds to the given angle $x$. (Using $x$ to stand for the measure of the angle can get confusing—remember that it does stand for the angle measure, in this case, and not for the corresponding $x-$coordinate.) The question this problem asks, therefore, is "What angle has the same $y-$coordinate as another angle that's twice as big?" Now, two angles can only have the same $y-$coordinate if they are the same angle, or if one of them is the reflection of the other across the $y-$axis. And the only angle whose reflection across the $y-$axis is also twice as big as itself is an angle of $\pi$ (or $-\pi$) radians (a rotation of half a circle). $\pi$ is outside the range of permissible answers we were given, so $-\pi$ is the solution we want. 0 is also a solution, because 0 doubled is still 0 and so an angle of measure 0 has the same coordinates as an angle twice as big.

*Solving Trigonometric Equations Using Half Angle Formulas*

**Half-Angle Identity for the Sine Function**

Basically, what we are doing here is applying the double-angle formula more or less in reverse to find the half-angle formula. Because we have to take the square root of both sides, however, we end up with two possible answers, one positive and one negative. We can tell which one is correct in any particular case, though, based on what quadrant the angle $\frac{\alpha}{2}$ is in: if it is in the first two quadrants, its sine must be positive, and if it is in the third or fourth quadrant, its sine must be negative.

(Similar reasoning is used below for the cosine half-angle formula.)

Example 1 should read "Use the half angle formula for the sine function to determine the value of $\sin 15°$" rather than $\sin 30°$.

# Equations with Inverse Circular Functions

*Solving Trigonometric Equations Using Inverse Notation*

**Introduction**

Students may be prone to mix up the ranges for the various inverse functions, and they may be baffled as to why the ranges are seemingly arbitrarily different. The rationale for choosing those particular ranges is that each of them represents two adjacent quadrants, in one of which the given function is positive and in the other of which it is negative. That way, the whole set of possible values for the trig function is covered, but none of the values are repeated.

For the cosine function, for example, it is convenient to pick the first two quadrants, because the graph of the cosine

function over the interval from 0 to $\pi$ hits every possible $y-$value exactly once. (The same holds true for the secant function, as it is the reciprocal of cosine.) But for the sine function, that interval won't work; the graph of the sine function from 0 to $\pi$ hits all the positive $y-$values twice, but none of the negative values. Instead, we use the interval from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$, or the fourth and first quadrants, because over that interval the sine function hits all the possible $y-$values once each. (We could also use the interval from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, or the second and third quadrants, but it's more convenient to use an interval that includes 0.) And of course we use that same interval for cosecant, the reciprocal of sine. (For the cosecant and secant functions, though, we have to leave out the point in the very center of the range where the sine or cosine equals 0 and the cosecant or secant is therefore undefined.)

Finally, for the tangent and cotangent functions, we choose an interval that includes all possible values of the function while avoiding the function's asymptotes; that is why the endpoints are excluded from the ranges of those inverse functions, while the other functions' ranges include their endpoints.

Note: In the graphs in this section, the degree signs should actually be $-1$[U+0080][U+0099]s; for example, [U+0080][U+009C]$y = sin°x$[U+0080][U+009D] should read [U+0080][U+009C]$y = sin^{-1}x$.[U+0080][U+009D]

### Points to Consider

We haven't yet covered any identities that will help us express inverse trig functions in other ways. However, the identities we've learned so far can help us when we encounter expressions that have both trig functions and inverse trig functions in them, as we can use them to simplify the non-inverse functions.

### *Solving Trigonometric Equations Using Inverse Functions*

### Points to Consider

This isn't a trick question; we can indeed use identities to solve trig equations, though we won't always be able to in any particular case.

### *Solving Inverse Equations Using Trigonometric Identities*

### Points to Consider

We will see the applicability to real-world problems in the following review questions. You might encourage students to think of some more cases where these techniques could be applied.

## 1.5  Triangles and Vectors

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### The Law of Cosines

**Introduction**

The definition of "oblique" (non-right) may be worth stressing briefly, as students may otherwise confuse it with "obtuse." (Adding to the confusion is the fact that obtuse triangles are also oblique triangles, although not all oblique triangles are obtuse.)

**Derive the Law of Cosines**

The fourth line of the derivation here contains a clever trick that is worth explaining. The text explains why we can substitute $a\cos C$ for $x$, but it may not be obvious why it is a good idea to do so. The reason is that it lets us express $c^2$ solely in terms of $a, b$, and $C$, so we now have a formula we can use for any triangle without needing to draw an altitude like BD again.

**Side of an Oblique Triangle (given the other two sides)**

"Note that the negative answer is thrown out as having no geometric meaning in this case" may bear explaining. The final step of example 1 involves taking the square root of both sides, which yields two possible answers, positive and negative. But we can safely disregard the negative answer because the number we are looking for represents the length of a line segment, which must be positive. This will be the case whenever we use the Law of Cosines.

Part 2 of the Real-World Application problem involves using the Law of Cosines differently, to find an angle when the sides are known instead of a side when the other two sides and an angle are known. This technique won't actually be explained until the next section, so you may need to walk through this example very carefully, or just skip ahead to the next section and then come back to it.

**Identify Accurate Drawings of General Triangles**

A problem like Example 3 could of course be done faster if we were in a hurry; we could simply skip to part 2, since finding out what the angle should be would also tell us if the angle given was the correct one.

**Points to Consider**

1) If we apply the Law of Cosines to a right triangle, the term $2ab\cos C$ becomes zero because $\cos 90°$ is zero. The Law of Cosines then reduces to the Pythagorean Theorem.

2) It is possible to solve the triangle completely through repeated applications of the Law of Cosines. Knowing two sides and the included angle lets us find the third side; then we can apply the theorem backwards, plugging in the three sides to find either of the missing angles; and then we can do this again to find the other missing angle, or simply find it by the Triangle Sum Theorem.

3) We cannot use the Law of Cosines if we only know one or no side lengths; if we know two side lengths but no angles; or if we know two side lengths and one angle, but the angle is not between the two sides. (It may seem that we could still apply the Law of Cosines in the latter case, but for some triangles it turns out that that yields two positive answers, and there is no way to tell which is correct. This is the Ambiguous Case described in a later lesson.)

4) Students may need to experiment a bit to answer this question. If they are stumped, remind them of the Triangle Inequality they may have learned in Geometry: the sum of any two sides of a triangle must be greater than the third

side. Any set of three numbers such that one of them is greater than the sum of the other two is a set of numbers that do not form a triangle.

## Area of a Triangle

**Derive Area $= \frac{1}{2}bc\sin A$**

The second line of the derivation here employs much the same trick as was used to derive the Law of Cosines in the previous lesson, and for similar reasons.

In estimating the cost of the pool cover here, we are simply calculating the exact cost by multiplying the precise area in square feet by $35, for simplicity's sake. For a small extra challenge, ask students to find the total cost if the price is a more realistic $35 per square foot or fraction thereof. (To solve this, they need to round up the area to the nearest whole number and then multiply by $35.) For a bigger extra challenge, ask them to find the cost if the cover of the pool needs to be 1 foot longer on each side than the pool itself. (To solve this, they would use the same formula as before, but $b$ and $c$ would be 25 and 27 instead of 24 and 26.)

**Find the Area Using Three Sides–SSS (side-side-side) Heron's Formula**

Observant students might worry that the terms (s-a), (s-b), and (s-c) in Heron's Formula could end up being negative, which could make the whole expression under the square root sign negative and make it impossible to find the square root. However, inspecting the formula for s shows us that the term (s-a) can only be negative if a is greater than b+c (similar reasoning holds for (s-b) and (s-c)), and the Triangle Inequality (mentioned in the notes to the previous lesson) guarantees that this can never be the case.

Incidentally, calculators will be needed throughout this chapter, as we are no longer working with special angles whose trig ratios we know, or even angles we can look up in tables like the one in chapter 1.

**Applications, Technological Tools**

The sailboat diagram may be somewhat confusing. The jib sail is the gray shaded area on the left-hand side of the mast, and the line labeled `[U+0080][U+009C]`$x$`[U+0080][U+009D]` is the rope attaching the sail to the mast.

**Points to Consider**

1) The Triangle Inequality is the answer to this question as well. If any one side were greater than half the perimeter, it would be greater than the sum of the other two sides, which the Triangle Inequality tells us is impossible.

2) In these three cases, we actually don't have enough information to solve the triangle or find its area using the techniques covered so far. We need to use the Law of Sines, which is the subject of the next lesson.

3) Applying Heron's Formula in reverse, though tedious, will yield the third side if the first two sides and the area are known.

## The Law of Sines

**Introduction**

Note that in the diagram accompanying the Real-World Application problem, A represents Chicago and $C$ represents Buffalo; make sure students don't get caught thinking $B$ is Buffalo and $C$ is Chicago.

The phrase "the side is not included" may also cause confusion. "Included" is being used here in the technical sense, so "not included" means that the side is not in between the two angles—even though the length of the side is clearly "included" in the set of information we are given.

**Derive Two Forms of the Law of Sines**

An interesting exercise is to apply the Law of Sines to a right triangle to verify its accuracy. If $a$ and $b$ are the legs of the triangle and $c$ is the hypotenuse, then $\frac{a}{\sin A} = \frac{a}{\frac{a}{c}} = c$. Similarly, $\frac{b}{\sin B} = \frac{b}{\frac{b}{c}} = c$, and $\frac{c}{\sin C} = \frac{c}{1} = c$ (because $\sin 90° = 1$). As predicted, the ratios of all three sides to the sines of their opposite angles are equal.

**AAS (Angle-Angle-Side)**

An isosceles triangle appears in the Real-World Application problem here; we know that sides $BC$ and $DC$ are congruent because the angles across from them are congruent. A question for students: how could the Law of Sines have told us this if we didn't already know it? (Hint: if $\frac{a}{\sin A} = \frac{b}{\sin B}$, and $A = B$ (so $\sin A = \sin B$), what does that tell us about $a$ and $b$?)

**ASA (Angle-Side-Angle)**

The ASA case is very close to the AAS case; in fact, by using the Triangle Sum Theorem to find the third angle, we are really turning the ASA case into the AAS case, because we now know two angles and a side that is not between them.

For students who are developing headaches trying to memorize all these three-letter acronyms, there is a simpler way to figure out when it is possible to use the Law of Sines. If you know two angles of a triangle, you really know all three angles, so if you know any two angles and one side, you can use the Law of Sines to find the other two sides.

**Applications**

The diagram in situation 2 is incompletely labeled : the top of the mountain should be marked $M$, the right angle $N$, and the $127°$ angle $U$. Side $u$ is then the same as side $x$.

**Points to Consider**

1) We still can't use the Law of Sines or Cosines if we don't know any of a triangle's side lengths, or if we only know one side and one angle. Also, we can't use them if we only know two sides and an angle that is not between them.

2) Two angles can have the same sine if they are each other's complements. This is why we can use the Law of Sines to solve for a side, but not for an angle; if we used it to solve for an angle, we might get two possible angles and not know which was correct.

3) With the Laws of Sines and Cosines together, we can solve any triangle if we know all three sides; two angles and one side; or two sides and the angle between them.

# The Ambiguous Case

**Introduction**

Note that the cases where the Law of Sines is useful are the cases where we know at least two angles. Thanks to the Triangle Sum Theorem, knowing two angles of a triangle really means we know all three angles, so the Law of Sines is useful when we know all the angles of a triangle (plus at least one side) and just need to find the remaining sides. It may therefore be useful to think "Use the Law of *Sines* to find the *sides*." (Just remember that this little saying doesn't always apply when we only know one angle—sometimes we need the Law of Cosines in that case.)

**Possible Triangles Given SSA**

In the case where $a < b$, we can see why comparing $a$ with $b \sin A$ tells us how many solutions there are if we think in terms of the Law of Sines. The Law of Sines tells us that $b \sin A = a \sin B$, and that equation in turn tells us the following things:

1) If a is **less than** $b\sin A$, $\sin B$ must be **greater than 1** in order to make $a\sin B$ equal to $b\sin A$. (If a by itself isn't "big enough" to equal $b\sin A$, it has to be multiplied by something greater than 1 to make the whole expression big enough.) But there is no angle whose sine is greater than 1, so there is no solution.

2) Conversely, if a is **greater than** $b\sin A$, $\sin B$ must be **less than** 1 in order to make $a\sin B$ equal to $b\sin A$. But for any given value of the sine function between 0 and 1, there are two different angles between $0°$ and $180°$ that correspond to it—one acute angle and one obtuse angle, which are each other's supplements. So there are two possible solutions.

3) Finally, if a is **equal to** $b\sin a$, $\sin B$ must equal 1 to keep the equation true—and there is only one angle between $0°$ and $180°$ whose sine is 1: a right angle. So there is one solution.

**Points to Consider**

1) One way to write the Law of Sines is $\frac{a}{b} = \frac{\sin A}{\sin B}$. This tells us that if a is greater than $b$, $\sin A$ is also greater than $\sin B$. The sine function is greater the closer an angle is to $90°$, so A must be closer to $90°$ than $B$.

- **If $A$ and $B$ are both less than $90°$**, this means that $A$ is greater than $B$.
- **If $A$ is $90°$ or greater**, we already know $A$ is greater than $B$ because only one angle in a triangle can be $90°$ or greater.
- **If $B$ is $90°$ or greater**, then $A$ being closer to $90°$ than $B$ means that $(90° [\text{U}+0080] [\text{U}+0093] A)$ must be less than $(B [\text{U}+0080] [\text{U}+0093] 90°)$, which means $A + B > 180°$—but that's not possible if $A$ and $B$ are two angles of a triangle, so that means $B$ just can't be $90°$ or greater.

So, the Law of Sines tells us that if $a > b$, $A > B$ and (more importantly) $B < 90°$. This means that if we apply the Law of Sines when $a > b$ and get two possible values for angle $B$, since one will be acute and one will be obtuse, we know that only the acute one can be correct.

(Applied more generally, the Law of Sines tells us that the biggest angle is opposite the biggest side and the smallest angle is opposite the smallest side. This is often useful to know.)

2) This was covered in the section above.

3) We can check whether each of the possible angles makes the total angle sum greater than $180°$. If it does, we throw it out; if not, it is a correct solution.

# General Solutions of Triangles

**Summary of Triangle Techniques**

Students should note that by applying the techniques in this table repeatedly, they can find *all* of the missing sides or angles in a triangle as long as they start out with enough information to find one of them.

**Using the Law of Sines**

In the Real-World Application problem, students are told to find the biggest angle first using the Law of Cosines in order to avoid dealing with the Ambiguous Case later when using the Law of Sines. This is because, as discussed in the notes to the previous lesson, only the biggest angle in a triangle can be obtuse, so finding the biggest angle first ensures that the other angles will be acute and will thus have only one possible value.

However, students should beware inaccurate diagrams! In a drawing that is not to scale (or one that *might* not be to scale), the angle that looks the biggest may not be the biggest. Instead of looking for what seems to be the biggest angle in the diagram given, if we know the exact side lengths we should make use of the fact that the biggest angle is opposite the longest side. (We can derive this fact from the Law of Sines, as described earlier.)

It may be best to skip the part that describes converting the angles to headings, as those headings can't be found accurately without knowing which way is due north .

**Points to Consider**

1) It's possible, but never really necessary, to use the Law of Sines before the Law of Cosines. There are times when we have enough information to use the Law of Sines but not the Law of Cosines, but in those cases, applying the Law of Sines always gives us enough information to finish solving the triangle without needing the Law of Cosines any more. Specifically, after applying the Law of Sines, no matter what information we started out with, we always end up knowing at least two sides and two angles, which means we really know two sides and three angles (by the Triangle Sum Theorem)—so one more application of the Law of Sines will give us the third side, and that's all we have left to find.

2) If we know three sides and one angle, we could apply the Law of Sines, but we might then find ourselves in the Ambiguous Case. To avoid that, we might prefer to use the Law of Cosines. In any other case where both laws are applicable, though, the Law of Sines is generally preferable simply because it is easier.

3) In both cases where the Law of Cosines is applicable, we end up knowing all three sides and one angle after we apply it. Therefore, we can then switch to the Law of Sines (SSA case), although as just mentioned, we may prefer to stick with the Law of Cosines instead.

# Vectors

**Introduction**

In the definitions of displacement, velocity, and force, the phrase "in a certain direction" is the important part. Displacement without direction is simply distance; velocity without direction is simply speed; and there is no special term for force without direction. These definitions are taken from the study of physics.

Since distance + direction = displacement and speed + direction = velocity, we can say that distance is simply the magnitude of the vector that represents displacement, and speed is the magnitude of the vector that represents velocity.

**Directed Line Segments, Equal Vectors, and Absolute Value**

Drawing the vector in example 1 is probably a good idea, as is reviewing the distance formula. Drawing a right triangle with the given vector as the hypotenuse will remind students of how the distance formula is derived from the Pythagorean Theorem. When students use the distance formula throughout the rest of this chapter, drawing right triangles will help them to remember it, but it is also a good idea for them to try to learn the formula well enough to use it without consulting diagrams.

The direction of a vector, by the way, is defined as the angle made by the vector when it is placed in standard position.

**Vector Addition and Subtraction**

You'll need to skip over the section on subtraction until after you've covered the section on addition just below it.

**Vector Addition**

Not only can we not use the parallelogram method to find the sum of a vector and itself, we also can't use it to find the sum of a vector and its opposite (and in the latter case, the tip-to-tail method doesn't work either.) However, the sum of a vector and its opposite is just 0, and the sum of a vector and itself is just a vector with twice the magnitude in the same direction.

**Resultant of Two Displacements**

When we return to the problem about the ship, you may need to remind students that the magnitude of each vector represents the speed the ship is traveling in that direction, and therefore that the total speed is represented by the magnitude of the resultant vector.

In the balloon example, angle $A$ is the "bottom corner" of the triangle in the diagram. There is an easier way to find

the angle with the horizontal, though, and you might ask students what it is. (Answer: The "top right corner" of the triangle is the angle we are looking for, and its tangent is $\frac{13}{22}$, from which we can calculate the angle directly.)

In case anyone is confused by part c of the "other things to consider" section, the phrase "(22ft/second times $14,400$ seconds in two hours)" should read "four hours." The numbers are still correct.

**Points to Consider**

When we add vectors using the triangle method, we know their lengths and can figure out the angle between them; hence we can use the Law of Cosines to figure out the magnitude and direction of the resultant vector.

This is a little less straightforward than it sounds, though, because we can't just read off the angle measures easily. Consider the following example, where $A$ and $B$ represent the directional angles of the vectors a and b that are being added by the triangle method:
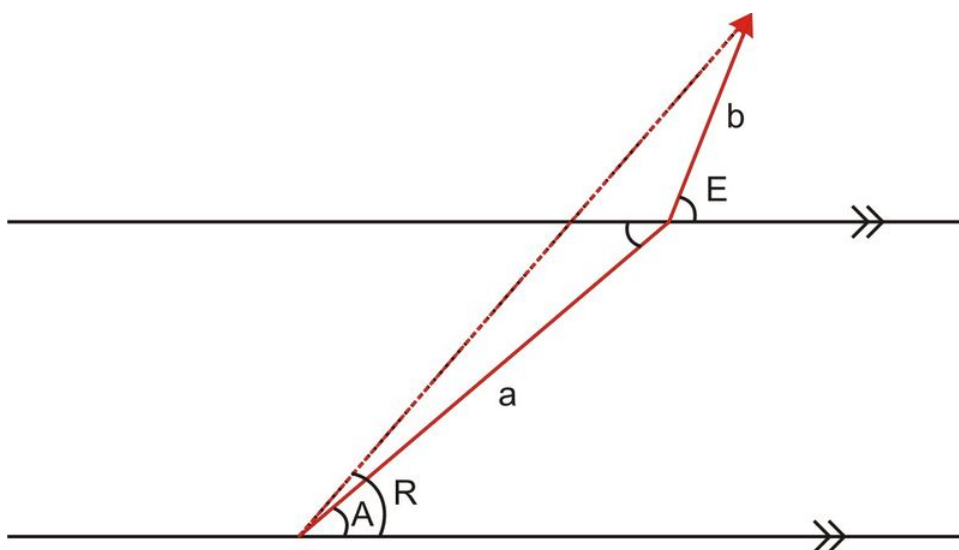


Unfortunately, $A$ and $B$ are the angles the vectors make with a horizontal line, not the angles within the triangle which we need to know in order to use the Law of Cosines—so first we need to use geometry to determine the angles within the triangle. We can see that the largest angle is equal to $A$ (by alternate interior angles) plus the supplement of $B$, and since we know sides $a$ and $b$, we can use the Law of Cosines to find the magnitude of the resultant vector. Then the Law of Sines will tell us the other angles of the triangle, from which we can figure out angle $R$, the direction angle of the resultant vector.

## Component Vectors

**Vector Times a Scalar**

Remind students here that $|\vec{a}|$ represents the magnitude of vector $\vec{a}$.

It may be worth explaining why we can "scale" a vector up or down just by multiplying each of its coordinates by the same scalar. Referring back to the methods from the previous lesson will show why this works: if you take the formula for finding the magnitude of the original vector, and multiply all the coordinates by the constant $k$, the result will be $k$ times the original magnitude; and if you use trig ratios to find the direction of the vector, those ratios will stay the same (and thus the direction stays the same) if you multiply all the coordinates by $k$.

**Translation of Vectors and Slope**

To multiply a vector by a scalar, as in Example 5, we don't actually need to translate it to the origin first; we can

just multiply the coordinates of the initial and terminal points of the vector by the scalar k. However, this problem provides useful practice in translating vectors as well as in scalar multiplication.

You may want to clarify that although the same ordered pair can represent many different vectors, it can only represent one vector in standard position. That vector is equivalent to, and in a sense represents, infinitely many other vectors with the same magnitude and direction but different initial points.

### Unit Vectors and Components

(Note: $\vec{i}$ and $\vec{j}$ are typically read as "$i-$hat" and "$j-$hat," but often you can call them simply `[U+0080][U+009C]`$i$`[U+0080][U+00` and `[U+0080][U+009C]`$j$`[U+0080][U+009D]` without confusing anyone.)

### Resultant as the Sum of Two Components

You may need to slow down to explain the notation here. In the diagram, the blue vector is $r$ and the green vector is $s$, but we are expressing them in a slightly different way to emphasize their role as component vectors. Since vector $r$ is horizontal, it can be expressed as $|\vec{r}|\hat{i}$—that is, as the horizontal unit vector $\hat{i}$ multiplied by the scalar quantity "the magnitude of $r$," yielding a vector with the same magnitude as $r$ (which makes sense because it is $r$!) and in the same direction as the horizontal unit vector $\hat{i}$. By the same logic, vector s can be expressed as $|\vec{s}|\hat{j}$, because it has the magnitude of $s$ and is in the direction of $\hat{j}$.

### Points to Consider

1) One way to verify answers to an addition or subtraction problem is to resolve each vector into its components and then add or subtract the components separately.

2) Vectors often form oblique triangles when added, so many of the same solving techniques apply.

3) Working with vectors can be more difficult because they are expressed in terms of their relationship to the origin rather than their relationship to each other, so finding things like the angles between them may be harder.

4) Vectors are usually used to solve problems where force is being applied in different directions, as opposed to problems that simply deal with distances between objects.

5) Unit vectors can help us visualize distances on a coordinate grid, but we don't generally need to use them when we are not looking for such distances.

## Real-World Triangle Problem Solving

### Introduction

To clarify situation 1: the climber will need enough rope to reach in a straight line from the top of the wall to the point where his partner will be standing (i.e. the point where he is now), so we need to find that distance to determine the necessary amount of rope.

To clarify situation 2: the axles run lengthwise through the centers of the cylinders and stick out at each end, so the steel cable to hold the cylinders together needs only to run around all three axles, rather than around the outsides of the cylinders. (This will be made clearer by the diagram in the next section.) Also, two loops of cable will need to be wrapped around the axles—one at each end of the cylinders—so once we find the length of one loop of cable, we will need to double it.

### Represent Problem Situations as Triangle(s)

In the diagram for situation 3, make sure students realize that the label θ applies to the whole span of the angle between the resultant vector and the horizontal line, not just the angle between the resultant vector and the vector next to it.

### Make a Problem-Solving Plan

In order to figure out what information they still need to solve each problem, students may first need to think about what tools they might use to solve it. Whether or not the triangle is a right triangle is especially important to consider, as right triangles can be solved with much easier methods.

**Choose Among All Available Tools**

In situation 1, note that we are rounding our answers to the nearest whole number of feet; this is reasonable since nothing more precise was specified. In situation 2, however, since we started off with values that were rounded to the nearest tenth, it is reasonable to round our answers to the nearest tenth as well.

**Points to Consider**

1) We could verify our answer in situation 1 using different trig ratios, or trig ratios of different sides, than we used to find the answer. In situation 2, we could use the Law of Cosines to verify the angles that we didn't find with the Law of Cosines in the first place; in situation 3, we could do that with the laws of both Sines and Cosines. Those would be less reliable methods, though, as we might get mixed up and simply perform the same operations we used to get the answers in the first place, thereby verifying nothing.

2) Checking your answer might not help if you verified the answer with the same methods you used to get the answer—or with those same methods in disguise.

3) The Law of Sines might be unreliable for checking angles because of the Ambiguous Case.

4) Some of the tools used to check them might have been used to solve them; also, applying the same tools in a different order might have worked.

# 1.6  Polar Equations and Complex Numbers

## Polar Coordinates

*Polar Coordinates*

**Introduction**

On example 1, you may need to stress that we are moving clockwise because the θ-coordinate is negative, and that otherwise we would be moving counterclockwise.

*Sinusoids of One Revolution*

**Introduction**

An important thing to explain about graphing in polar coordinates is that $r$ is always the dependent variable and θ is the independent variable—so the equations we graph in polar coordinates will take forms like $r = \sin\theta$, or more complex expressions based on θ, where θ is always an angle measure. The fact that $r$ is always written first in the ordered-pair representation of a point is a little counterintuitive, because we have previously been used to seeing the independent variable (which is usually $x$) written first.

Why do we write $r$ first if $r$ is the dependent variable? One reason is that graphing points is easier, or at least more intuitive, if we look at the $r-$coordinate first; we can think of ourselves as starting at the origin, moving along the $x-$axis to the given $r-$value, and then moving around the circle to the given θ$-$value.

(Another reason has to do with the conventions for representing complex numbers in polar form, which will be covered later in this chapter.)

Example 3 contains an interesting optical illusion: the diagonal lines passing through the graph make it look slightly warped and may prevent students from realizing that it is in fact a perfect circle.

Also worth pointing out about this example is that all of the points on the graph have in fact been traced twice over. The θ$-$values from $0°$ to $180°$ traced out the circle; then the θ$-$values from $180°$ to $360°$ produced the same set of $r-$values all over again, but negative. Since $(-r, \theta + 180°)$ always represents the same point as $(r, \theta)$, the second set of $r-$values correspond to the same set of points as the first set.

The important thing to point out about cardioids is that they are dimpled limaçons whose "dimple" passes directly through the pole. The depth of the dimple in a limaçon depends on the ratio $\frac{a}{b}$ (the smaller the ratio, the deeper the dimple), and the limaçon becomes a cardioid when the ratio equals 1. If the ratio got any smaller, the limaçon would dimple so far that it would develop a loop.

*Applications, Trigonometric Tools*

**Introduction**

The constraints on example 1 should read "$-2\pi \le \theta \le 2\pi$[U+0080][U+009D] rather than [U+0080][U+009C]$0 \le \theta \le 2\pi$[U+0080][U+009D].

# Polar-Cartesian Transformations

*Graphs of Polar Equations*

Note that when we graph a basic cosine equation in polar coordinates, the domain only needs to go from $0$ to $\pi$ rather than $2\pi$. As noted in the previous lesson, the sine or cosine graph is traced out twice over on an interval of $2\pi$ units. In a sense, we can almost say that when we are using polar coordinates, the sine and cosine functions have a period of $\pi$ instead of $2\pi$.

**Conic Section Transformations**

**Introduction**

An ellipse is actually the result of the intersection of a cone on *one* side by a plane that may or may not be parallel to the base of the cone. When the plane is parallel to the base, the ellipse is a circle.

The plane that creates a parabola cannot just be non-parallel to the base; it must be parallel to the slanted line that forms the edge of the cone.

The plane that creates a hyperbola does not actually have to be perpendicular to the base, as long as it intersects both halves of the cone.

Also, you may need to clarify that the definition of "cone" used here differs from the one students encountered in geometry: a "cone" here is really two cones lined up tip to tip, and the two cones actually extend infinitely far outward from the point where they meet.

Parabolas can theoretically be stretched horizontally as well as vertically, but stretching them horizontally is in a sense the same as shrinking (or "un-stretching") them vertically, so it can be expressed in the same way, with a little adjusting of arbitrary constants.

The focal axis of an ellipse is also called the major axis, and its length is denoted as 2a. The perpendicular line passing through the center of the ellipse is the minor axis and its length is 2b. This will be important later.

**Points to Consider**

Circles centered at the origin are certainly easier to express in polar coordinates, but those that have been shifted away from the origin may be a little harder. Parabolas tend to be easier to represent in rectangular coordinates. In general, taking a familiar equation and shifting or stretching it in one direction or another tends to be easier when the equation is expressed in rectangular coordinates. (These are just a few examples; students may provide others.)

Polar curves may of course intersect, as we saw during this lesson. This question prepares students for the next lesson, where they will learn to find the intersection points of such curves.

Since two different sets (in fact, infinitely many sets) of coordinates can be used to represent the same set of points, it makes sense that two different equations could be used to represent the same polar curve. For example, $r = \sin\theta$ would produce the same graph as $r = \sin(\theta + 2\pi)$.

One important difference between rectangular and polar representation is that polar graphs are more likely to stay within a finite viewing space. When we graph a function on a rectangular grid, if the function's domain is unlimited, then the graph extends infinitely far to each side, so we can't ever really draw the entire graph. Polar graphs, on the other hand, can extend infinitely far outward if the range is unlimited, but if the range is limited, then the domain can be unlimited and the graph will still be conveniently compact.

(Then again, this can be an inconvenience at times, as it makes it harder to show when we are deliberately only graphing part of a function instead of the whole thing. Since the whole thing *should* fit in the visible part of the graph, viewers will expect the visible part of the graph to contain the whole function unless we include a note specifying otherwise. With rectangular coordinates, we can simply narrow the graphing window to show only the part we want to graph, and viewers don't need to be told that there's really more to the graph than just that part.)

## Systems of Polar Equations

*Graph and Calculate Intersections of Polar Curves*

### Introduction

In the solution to example 2, the notation $k\varepsilon I$ may be unfamiliar to students; it means "$k$ is an element (a member) of the set of all integers," or simply "$k$ is an integer." So saying that the solution set includes $\left(1, \frac{5\pi}{3}\right) + 2\pi k, k\varepsilon I$ is simply another way of saying that when we add any integer multiple of $2\pi$ to the $\theta-$coordinate in the solution $\left(1, \frac{5\pi}{3}\right)$, we get another valid solution—and of course the same holds true for the solution $\left(1, \frac{\pi}{3}\right)$.

The solution to Example 3 may be confusing at first—how can $(0,0)$ and $\left(0, \frac{\pi}{2}\right)$ represent the same point? Students should grasp by now that adding a multiple of $2\pi$ to the $\theta-$coordinate of a point yields another representation of the same point, but in this case we've added $\frac{\pi}{2}$, which isn't a multiple of $2\pi$—so what's going on here?

The important thing to explain here is that the pole (as we call the origin when we are using polar coordinates) is a very special point. Normally, any given $r-$coordinate designates a circle centered at the pole with radius $r$, and we use different $\theta-$coordinates to pick out specific points on the circle. But a circle with radius 0 is just a single point—the pole itself—so no matter what $\theta-$coordinate we choose, we always end up at that same point. $(0,0)$ represents the same point as $(0,3), \left(0, \frac{\pi}{2}\right), (0,4\pi)$, or any ordered pair whatsoever that has 0 as the $r-$coordinate.

### Points to Consider

A very simple example of two polar curves that do not intersect is the pair $r = 1$ and $r = 2$. And we have seen again in this lesson how the same point can be expressed in more than one way in polar coordinates; in the next lesson we will see how the same curve can too.

*Equivalent Polar Curves*

It is important that students do not get the mistaken idea that expressions with equivalent graphs are necessarily equivalent expressions. In example 1b, the two equations graphed are indeed equivalent, as they are both simply different ways of expressing $r = 5$. But in example 1a, although the two equations trace out the same graph, they do not actually have the same $r-$value for any given $\theta-$value, and so are not equivalent equations. If they were equivalent, plugging the same $\theta-$value into both of them would always yield the same $r-$value.

This is especially confusing because it only happens in polar coordinates, where the $\theta$-values overlap and repeat themselves. Here's an analogy that may help you to explain it: Suppose you ride the same bus to work or school every day, and suppose the bus maintains a very strict schedule, so it always reaches the same stop on its route at exactly the same time each day. (Of course no real bus could manage this, but let's assume it does in order to simplify the problem.) Now suppose you draw a graph each day representing the route the bus travels, and when you compare two consecutive graphs, you see that they look exactly the same—if you plotted them on the same axes, they would look like just one graph. But does this mean they are describing the exact same trip? No—they represent two different trips taken on two different days, and when you plot them on the same axis you are simply leaving out the "two different days" part. Really, the time-values of the second graph are the time-values of the first graph "plus 24 hours," and if you choose to plot them on the same graph to save space, you must still remember that the times on the two graphs aren't "really" the same.

And that's what happens when we use polar coordinates—we sometimes end up graphing the $\theta-$values for two or more different $r-$values in what looks like the same spot, but we must remember that just because they are sharing space, that doesn't mean they are really the same coordinates. Even when a whole graph looks the same as another, sometimes it simply consists of a different set of values that happen to be graphed in the same places.

# Imaginary and Complex Numbers

*Recognize*

**Introduction**

Here's another example of why the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ only applies if $a$ and $b$ are not both negative: Without that exception, we could apply the rule to $\sqrt{36}$ and express it as $\sqrt{-4}\sqrt{-9}$, which would equal $2i$ times $3i$, or $-6$. Technically, $(-6)^2$ is of course 36, but officially -6 is not the square root of 36, so that answer would be incorrect.

The last line of the solution to example 2b should have a 5, rather than a 3, under the radical sign.

**Points to Consider**

Students needn't know the answers to these questions; they are simply a preview of the next section.

*Standard Form of Complex Numbers (a + bi)*

**Introduction**

Students may be a little confused by the statement that a and b are both real numbers in the standard form $a + bi$. Clarify if necessary that the imaginary part of a complex number is $bi$, not just $b$; $bi$ is a pure imaginary number because it is a real number multiplied by $i$.

Students may not quite see why the answer to example 1c is expressed as it is. The answer is indeed in standard rectangular form, with $a = 3\sqrt{2}$ and $b = -2\sqrt{2}$, but we traditionally put the $i$ in front of the radical sign so that it doesn't look like it is included under the radical sign, and that makes it harder to see that the whole expression has the form $a + bi$.

After reviewing example 3, you might also want to challenge students to find the conjugate of a real number, like 5. (Answer: 5 is really $5 + 0i$, so its conjugate is $5[U+0080][U+0093]0i$, or simply 5 again. In other words, the conjugate of any real number is simply itself.)

**Points to Consider**

We will see in the next lesson what operations can be performed on complex numbers and with what results.

*Complex Number Plane*

If you've covered vectors recently, you might point out here that the absolute value of the complex number $a + bi$ is the same as the magnitude of the vector represented by the point $(a, b)$.

You may need to skip over the problem about two students walking home, since the original formulation of the problem is missing from the lesson.

# Operations on Complex Numbers

*Quadratic Formula*

**Points to Consider**

When the roots of an equation are complex, we know that the graph of the equation does not intersect the $x-$axis. Conversely, when the graph does not intersect the $x-$axis, we know the roots are complex, and when it does, we know there are either two real roots or one real root repeated twice.

*Sums and Differences of Complex Numbers*

**Points to Consider**

We will see in the next lesson how complex numbers can be expressed in polar form.

*Products and Quotients of Complex Numbers (conjugates)*

**Introduction**

There is a typographical error in the formula for multiplying complex numbers: the term that reads $(ad - bd)$ should read $(ac - bd)$.

Students may not quite see where the [U+0080][U+009C] $- bd$ [U+0080][U+009D] term comes from. Explain that multiplying *bi* and *di* yields $bdi^2$, and $i^2$ is simply $-1$, leaving us with [U+0080][U+0093]$bd$.

The procedure for dividing complex numbers may make more sense if you remind students that *i* is equal to $\sqrt{-1}$. When we express the quotient of two complex numbers as a fraction, substituting $\sqrt{-1}$ for *i* shows that this fraction essentially has a radical in the denominator which we must rationalize. Multiplying the numerator and denominator by the conjugate of the denominator, then, is clearly the way to get the imaginary part out of the denominator; and once the denominator is a real number, we can divide both parts of the numerator by that real number and thus express the answer in standard form.

**Points to Consider**

Once again, we will see in the next lesson that the answer to both of these questions is "yes."

*Applications, Trigonometric Tools*

**Operations on Complex Numbers**

Example 2 provides an excellent illustration of the relationship between operations on complex numbers and operations on vectors. You may want to point out to students that when they solve a problem like this by working with the real and imaginary parts of complex numbers separately, they are really just resolving vectors into horizontal and vertical components and working with each component separately. But instead of using $\hat{i}$ and $\hat{j}$ to represent those components, they are using 1 and *i*, because the horizontal component is a multiple of the unit vector in the real direction or "$1-$direction" and the vertical component is a multiple of the unit vector in the imaginary or "*i*[U+0080][U+0093]direction."

# Trigonometric Form of Complex Numbers

*Trigonometric Form of Complex Numbers: Relationships among $x, y, r$, and $\theta$*

In case students don't immediately see why $x = r\cos\theta$ and $y = r\sin\theta$, you can show them fairly easily on the diagram that $\sin\theta = \frac{y}{r}$, and then solving for *y* yields $y = r\sin\theta$. Similar reasoning, of course, holds for *x*.

*The Trigonometric or Polar Form of a Complex Number* $(rcis\theta)$

The term [U+0080][U+009C]$cis\theta$[U+0080][U+009D] is easier to remember if you point out that it is somewhat like an acronym, derived from [U+0080][U+009C]$\cos\theta + i\sin\theta$.[U+0080][U+009D]

The term "argument" is also worth explaining, as it often appears elsewhere in mathematics. Generally it refers to the "input" of a given function, so for example, in the expression [U+0080][U+009C]$\cos\theta$, [U+0080][U+009D]$\theta$ is called the argument of the cosine function. In the polar form of a complex number, of course, $\theta$ appears as the argument of both the sine and cosine functions, so it makes a kind of sense to call it the "argument" of the complex number as a whole.

Thinking of r as the "absolute value" of a complex number may be counterintuitive for students, but it is really just an extension of the idea of absolute value of real numbers. The absolute value of a real number is its distance from 0; the absolute value of a complex number is its distance from the point $(0,0)$. And in the complex plane, the distance of a real number from 0 becomes the same thing as its distance from $(0,0)$.

*Trigonometric Form of Complex Numbers: Steps for Conversion*

**Introduction**

*1.6. POLAR EQUATIONS AND COMPLEX NUMBERS*

The very first table in this section is the most useful for students to know, and it is most helpful for them to think of the left half and the right half separately. Emphasize that if they know the polar coordinates $r$ and $\theta$, they should use the two equations on the *left* to find the rectangular coordinates $x$ and $y$, whereas if they know $x$ and $y$ they should use the two equations on the *right* to find $r$ and $\theta$. (They *could* use the equations on the right to get $x$ and $y$ from $r$ and $\theta$, or use the equations on the left to get $r$ and $\theta$ from $x$ and $y$, but that would be a much messier process.)

The last step of Example 3 is important: finding the inverse tangent just tells us the reference angle for $\theta$, not $\theta$ itself. We must then apply our knowledge of what quadrant the complex number is in to figure out what angle $\theta$ really is. We can find out what quadrant the number is in by graphing the rectangular coordinates we started out with, or by simply noting the signs of those coordinates: $x$ is only positive in the first two quadrants, and $y$ is only positive in the first and fourth.

**Points to Consider**

In the next lesson, we will see how to perform basic operations such as multiplication and division on complex numbers in polar form.

## Product and Quotient Theorems

### *Product Theorem*

The first line of the derivation here uses the FOIL method for multiplying binomials; in the second line, we group together the terms with $i$ in them and factor out the $i$; and in the third line, we apply the angle sum rules for sine and cosine in reverse.

The Product Theorem may be easier for students to remember if summarized in words: "To multiply complex numbers in polar form, multiply the $r-$coordinates and add the $\theta-$coordinates."

### *Quotient Theorem*

Similarly, the Quotient Theorem can be summarized: "To divide complex numbers in polar form, divide the $r-$coordinates and subtract the $\theta-$coordinates." The derivation follows much the same procedure as the one for the product theorem; note that the denominator of the final fraction equals 1, so we can cancel it out.

Incidentally, some students may notice that we haven't discussed how to add and subtract complex numbers in polar form. It turns out that there is no handy formula for doing so; the only good way to add and subtract complex numbers is to convert them to rectangular form first.

### *Using the Quotient and Product Theorem*

**Introduction**

Students may be confused by example 3, or may be tempted to do the division on the numbers as given, in rectangular form. Doing it as the book suggests, however, will give them practice in converting from rectangular to polar form as well as in performing division on numbers in polar form. Meanwhile, the other three examples will give them practice in working with complex numbers expressed in polar form in more than one way.

For extra practice, you might have them find both the product and the quotient of the two numbers in each example, instead of just the product or the quotient.

**Points to Consider**

So far, we've actually just barely touched on squares and square roots of complex numbers, and we still don't know how to find them for most numbers. The next lesson will cover these as well as other powers and roots of complex numbers.

### *Applications and Trigonometric Tools: Real-Life Problem*

You may need to skip these problems, as they contain terms that students are not likely to know .

## Powers and Roots of Complex Numbers

### *De Moivre's Theorem*

A more intuitive way to express De Moivre's Theorem is "To raise a complex number to the $n^{\text{th}}$ power, raise the $r-$coordinate to the $n^{\text{th}}$ power and multiply the $\theta-$coordinate by $n$."

In example 2, the expression $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\right)$ should read $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)$.

### nth Root Theorem

Here's a much more intuitive way to explain the $n^{\text{th}}$ Root Theorem:

Every complex number has exactly $n$ $n^{\text{th}}$ roots, which are evenly spaced around a circle in the complex plane. If the original number has coordinates $(r, \theta)$, then the first of the $n^{\text{th}}$ roots (which, incidentally, is known as the *principal root*) has coordinates $\left(\sqrt[n]{r}, \frac{\theta}{n}\right)$. The rest of the $n^{\text{th}}$ roots all have the same $r-$coordinate, and their $\theta-$coordinates are each $\frac{\theta}{n}$ plus some multiple of $\frac{2\pi}{n}$; in other words, each of them is $\frac{1}{n}$ of the way around the circle from the one before it.

For example, the fourth roots of $(16, 60°)$ are $(2, 15°), (2, 105°), (2, 195°)$, and $(2, 285°)$. Plotting these points shows that they are evenly spaced around a circle of radius 2, and a little thought will show why. Raising 2 to the fourth power of course gives us 16, and multiplying an angle of 15° by 4 gives us 60° —but multiplying an angle of 15°-plus-some-multiple-of-90° by 4 gives us 60°-plus-some-multiple-of-360°, which is equivalent to 60°. So that's why, if $(2, 15°)$ is one fourth root of $(16, 60°)$, all the other fourth roots are of the form $(2, 15° + k * 90°)$ for some integer $k$. (If they were fifth roots, they'd be $\frac{360°}{5}$ or 72° apart; if they were sixth roots they'd be 60° apart, and so on.)

Incidentally, we could just keep going, adding 90° to each previous $\theta-$coordinate to get yet another one. But of course, after we've added 90° three times to the first solution to get three more solutions, adding 90° one more time would just give us the first solution plus 360°, or in other words the first solution all over again, and then we'd start cycling through all the solutions over again. So the first four solutions (the first $n$ solutions, if we're finding $n^{\text{th}}$ roots) are the only unique ones, and we can stop after finding them.

### *Solve Equations*

It should become clear here that we can use polar coordinates to determine the $n^{\text{th}}$ roots of a pure real number or pure imaginary number just as easily as a complex number. However, the roots are usually complex and tend to be somewhat messy to express in rectangular coordinates. For example, in the problem shown in the text, we can see that the roots can be expressed precisely in polar coordinates, and that we can use just one expression to summarize them all, whereas in rectangular coordinates we must list all the roots separately and can only express them in decimal approximations.

# CHAPTER 2 Trigonometry TE - Common Errors

## CHAPTER OUTLINE

# 2.1 Trigonometry and Right Angles

This Trigonometry Teaching Tips FlexBook is one of seven Teacher's Edition FlexBooks that accompany the CK-12 Foundation's Trigonometry Student Edition.

To receive information regarding upcoming FlexBooks or to receive the available Assessment and Solution Key FlexBooks for this program please write to us at teacher-requests@ck12.org.

## Basic Functions

**In-Text Examples**

1) Students may get mixed up and think that a relation is a function if it has a unique $x-$value for every $y-$value instead of the other way around. Stress that if the same $x-$value shows up twice, paired with two different $y-$values, then the relation is not a function (so relation A is not), but if every $x-$value has only one corresponding $y-$value, the relation is a function (so relation B is).

2) On part a, students may think the domain is restricted to positive numbers; explain that that restriction applies to the range, not the domain. (We can find the square of any real number $x$, but the $y-$value we get back will always be positive.)

On part c, students might think of trying to draw a line or curve connecting those points, or might just interpolate to assume that the domain includes all the numbers between 2 and 5. Explain if necessary that those points aren't representative samples of the function, they are all the points in the function, so the domain consists only of the three $x-$values given.

4) Students are likely to try plugging the numbers 15 and 30 into their equations; this will lead to error because those are the times in minutes, not hours, while the speeds given are in miles per hour. Remind them to convert minutes to hours before solving the problem.

6) Of course, the most common mistake here is to forget that only three sides of the enclosure need to be fenced because the barn will make the fourth side.

**Review Questions**

1) Part a is subject to the same error as Example 1 above. For students stuck on part b, suggest drawing a graph, or just ask them to consider whether it seems possible to plug in the same $x-$value twice and get two different $y-$values.

3) Part d is a possible sticking point, as students may not know what to do with the equation from part c. Remind them that they need their profit to be greater than zero, so they must find the smallest value of $x$ that will make $P(x)$ greater than zero. Also, remind them that their answer should be a whole number.

4) On part b, students may jump to the conclusion that the range is just the positive real numbers, because the equation resembles $y = x^2$. When they graph the function, point out to them that part of the graph is below the $x-$axis, so the range actually includes those $y-$values as well. Specifically, since the vertex has $y-$coordinate $-3.25$, the range consists of all real numbers greater than or equal to $-3.25$.

5) Students may identify the vertical asymptote at $x = -3$ but miss the horizontal one at $y = 1$, especially because it's not as easy to derive from the equation.

6) Students may get confused and set up the equation as if the cost per person were $500 and they were trying to

figure out the total cost for $p$ people. Stress that \$500 is the total cost and that they are trying to figure out how much each person will pay if the bill is split among $p$ people.

# Angles in Triangles

**In-Text Examples**

1) Students shouldn't get hung up on trying to figure out if the triangles are right, acute, or obtuse; it's harder to figure those out from the side lengths alone, but easy to tell if the triangle is equilateral, isosceles, or scalene. They may notice that $3-4-5$ is a Pythagorean triple (making that triangle a right triangle), and it may be worth pointing out that all equilateral triangles are acute (because all three of their angles must measure $60°$).

(If they really want to know how to tell if a triangle is acute or obtuse just from the side lengths, you can tell them this: Label the longest side $c$ and the other two sides $a$ and $b$, and then compare $c^2$ to the quantity $(a^2 + b^2)$. If $c^2$ is greater than the sum of the other two squares, the triangle is obtuse; if it is smaller, the triangle is acute; and if it is equal, of course, the triangle is right.)

2) Students should avoid jumping to the conclusion on part c that the $50°$ angle is one of the two angles that are equal—although it could be, it might not be, and so there are two possible solutions.

**Review Questions**

1) This problem has two right answers, so don't let students agonize over which one is correct.

4) For part a, students may need to be reminded what a complement is. (Complementary angles are angles that add up to $90°$.) Then, they may try to solve part b by working out $180 - 90 - 23 = 67$. This method does yield the correct answer, but misses the point of the problem: once we know the two non-right angles are complementary, all we have to do is subtract 23 from 90 to get the missing angle.

5) Drawing a picture may help students who get stuck on this problem, even though the problem is really more algebra than geometry. The important thing they may forget is that they know all three angles must add up to $180°$, which means they can set up the equation $D + O + G = 180$, substitute 2D and 3D for $O$ and $G$ respectively, and solve for $D$.

8) The triangles certainly look similar, and students may think they are because they have two sides in proportion and one angle the same. However, since the angle isn't between the two sides, we can't actually tell if the triangles are similar.

9) The fact that the numbers 100 and 20 appear next to each other may tempt students to set up the proportion $\frac{20}{100} = \frac{24}{x}$, or $\frac{100}{20} = \frac{24}{x}$. Have them draw a diagram to see which distances they are actually comparing, or simply remind them that they must pair the flagpole with *its* shadow and the building with its shadow to get the right proportion: $\frac{20}{24} == \frac{x}{100}$.

10) Make sure answers to this problem demonstrate an understanding that similar triangles are not necessarily congruent—i.e. they do not necessarily have the same side lengths.

# Measuring Rotation

**In-Text Examples**

1) For students who have trouble keeping the terms "acute" and "obtuse" straight, the mnemonic "a cute little angle" may help remind them that acute angles are the smaller ones.

2) Since protractors like the one shown here display two different numbers for the measure of an angle, students may

read off the wrong one. Remind them to think about whether the angle appears to be acute or obtuse, and figure out which number makes sense based on that. (Another way to check is to note that the end of the protractor they placed against one side of the angle either reads 0° on the inside "track" and 180° on the outside track, or vice versa. The track on which it reads 0° is the one from which they should read off the measure of the other side of the angle. For example, in the illustration shown in the text, the end of the protractor placed against the bottom side of the angle reads 0° on the inside track, so the inside track is the one to use for determining the angle measure, and so we know it is 50° and not 130°.)

3) Students may get their calculations backwards here, possibly due to a vague notion that the larger wheel should rotate a greater number of times. Explain if necessary that since the wheels both travel the same distance along their circumference and the larger wheel has more circumference, it doesn't have to make as many rotations to travel that distance. (A possibly useful analogy is that of a shorter person having to take more steps to keep up with a taller person.)

**Review Questions**

3) On part c, students may think they are done when they have converted the decimal portion to minutes and forget about converting the remainder to seconds, or they may "convert" to seconds by just copying the number after the decimal point—e.g., expressing $57.6 \texttt{[U+0080][U+0099]}$ as $57 \texttt{[U+0080][U+0099]} 6 \texttt{[U+0080][U+009D]}$. Remind them that $57.6 \texttt{[U+0080][U+0099]}$ is equal to $57 \frac{6}{10}$ minutes, and they need to figure out how many seconds are in $\frac{6}{10}$ of a minute if each second is $\frac{1}{60}$ of a minute.

4) Possible errors here include converting the seconds but not the minutes to decimal form, or the minutes but not the seconds. Referring back to page 33 should help students remember how to perform these conversions.

7) Students may get the diameter of the wheels mixed up with the distance between them, and plug in the wrong one at the wrong time.

9) $-120°$ is a possible wrong answer for part a. Demonstrate that an angle of $-120°$ falls in quadrant III, while an angle of 120° falls in quadrant II, so they are not co-terminal.

10) The length of the axles is a red herring, and so is the distance between them; students may think they need to find which one of the four wheels makes the most rotations, but really they only need to find whether the front or back wheels rotate more. (Also, the answers given in the text are incorrect; the numbers of revolutions should be 200 and 66.67 respectively, and the difference in the number of degrees should be 48000.)

**Additional Problems**

1) What is the angle between the hands of a clock at 6:30? (Remember, the hour hand is not directly on the 6.)

2) Name an angle that is coterminal with $-180°$.

**Answers to Additional Problems**

1) 15°

2) Answers will vary. Possible answers include 180° and 540°.

# Defining Trigonometric Functions

**In-Text Examples**

3) Students may mix up the definitions of secant and cosecant. Emphasize that secant is the reciprocal of *co*sine and *co*secant is the reciprocal of sine, so there is exactly one "co-" function per pair of reciprocals (and since tangent and cotangent are reciprocals, they too fit this pattern).

5) Students who are still thinking in terms of angles in triangles may get stuck here. Remind them to think of the trig functions as ratios of $x-$ and $y-$coordinates instead; using the definitions above, they can plug in any values for

*2.1. TRIGONOMETRY AND RIGHT ANGLES*

*x* and *y*, even values for which it isn't possible to draw a triangle and measure the side lengths.

Also, because students first learned the definition of sine before the definition of cosine, they can easily get confused and think that the sine value is the $x-$coordinate and the cosine value is the $y-$coordinate. Even if they "know" that's wrong, it's still an easy trap to fall into any time they're not thinking very hard about it (sometimes even after they've been studying trigonometry for quite a while!). Remind them to watch out for this error and to double-check their answers whenever they are finding sines and cosines with this method.

### Review Questions

6) The answer to part b, of course, is not simply $2y$; it's true that the length of *BD* is $2y$, but the point is that it is also 1 because triangle *ABD* is equilateral. Similarly, the answer to 6c is not just *y*, but $\frac{1}{2}$; this means that $y = \frac{1}{2}$, and that's important for solving the rest of the problem.

8) The ratios for $60°$ angles are easy to mix up with the ratios for $30°$ angles, especially since the values of a given trig function for a $60°$ angle is the same as the value of the corresponding "co-" function for a $30°$ angle.

9) Students may give the knee-jerk answer "quadrants I and II" because that's where the $y-$values are positive, or "quadrants I and IV" because that's where the $x-$values are positive. Remind them that the value of the tangent function depends on both the $x-$ and $y-$value: since the tangent is $\frac{y}{x}$, is it positive or negative when *x* and *y* are both positive? Both negative? How about when one is positive and the other is negative? In which quadrant(s) does each of those conditions hold?

10) Possible wrong answers include "it's five times $30°$," "it's the supplement of $30°$," and "it's $30°$ plus $120°$." Although these are all technically true, they aren't what we're looking for because they aren't useful in this case. The correct answer is along the lines of "it's like a $30°$ angle, but reflected across the $y-$axis," because noticing this fact helps us figure out what the ordered pair for a $150°$ angle is.

### Additional Problems

1) Sketch the angle $210°$ on the unit circle. What do you think its ordered pair is?

### Answers to Additional Problems

1) $\left( \frac{-\sqrt{3}}{2}, \frac{-1}{2} \right)$

# Trigonometric Functions of Any Angle

*In-Text Examples*

1) Make sure students remember that the reference angle is always the distance to the closest part of the $x-$axis, never the $y-$axis—even if the $y-$axis is closer.

4) Students may be momentarily thrown by part b; remind them that an angle of *negative* $300°$ is co-terminal with an angle of positive $60°$.

5) Some students may try dividing the given angles by some number, instead of subtracting $360°$ from them. Also, on later problems they may forget that they aren't done when they reduce the angle down to one that's less than $360°$, and that they still have to find the reference angle for that angle.

6) Note that the cosine column comes before the sine column; this may confuse students momentarily.

7) Make sure calculators are in degree mode; in radian mode the answer will appear to be 0.8589.

### Review Questions

7) "Between 10 and 15 degrees" or "between 165 and 170 degrees" is as precise as the answer needs to be. (Either of those approximations is correct, and students should be made aware of this, as the fact that there is more than one

angle for a given sine value will be important later.)

8) Students may think of choosing $60°$ as the "special angle." The value of $\tan(50°)$ is fairly close to the value of $\tan(60°)$, but you'd need a calculator to figure out the value of $\tan(60°)$ in decimal form in order to compare the two tangent values. The value of $\tan(45°)$, though, is simply 1, and in any case $50°$ is closer to $45°$ than to $60°$, so it makes more sense to use $45°$ as the special angle.

9) Leaving calculators in radian mode will yield the wrong answers $-0.9820$ and $45.1831$.

10) Students may end up thinking a little too hard about this problem. All they're supposed to conjecture is that the two expressions are not equal, so if any of them struggle with this problem, find out if they've got that much figured out and reassure them they can stop there.

### Additional Problems

1) Use a calculator to find the tangent of $86°, 87°, 88°,$ and $89°$. Then, find the tangent of $94°, 93°, 92°,$ and $91°$. Now make a conjecture about the behavior of the tangent function as $x$ approaches $90°$.

### Answers to Additional Problems

1) The values of the tangent function are as follows:

| $X$ | $\tan x$ |
|---|---|
| $86°$ | $14.3007$ |
| $87°$ | $19.0811$ |
| $88°$ | $28.6363$ |
| $89°$ | $57.2900$ |
| $90°$ | undefined |
| $91°$ | $-57.2900$ |
| $92°$ | $-28.6363$ |
| $93°$ | $-19.0811$ |
| $94°$ | $-14.3007$ |

The tangent function approaches infinity as $x$ approaches $90°$ from below, and approaches negative infinity as $x$ approaches $90°$ from above.

## Relating Trigonometric Functions

### In-Text Examples

4) Demonstrating that $\cot\theta = \frac{\cos\theta}{\sin\theta}$ because cot is the reciprocal of tan and $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is an equally valid answer.

5) Some students may try to simply subtract $\cos\theta$ from 1 to get $\sin\theta$; others may subtract $\cos^2\theta$ from 1 but then forget to take the square root of the answer.

6) The biggest problem here is that students may simply not have any idea where to begin. Telling them the first step may help, but it may also be easier for them to work the problem out "backwards" instead. When you start with $\cot^2\theta + 1 = \csc^2\theta$, the logical first step is to rewrite $\cot^2\theta$ as $\frac{\cos^2\theta}{\sin^2\theta}$ and $\csc^2\theta$ as $\frac{1}{\sin^2\theta}$; this suggests the idea of also writing 1 as $\frac{\sin^2\theta}{\sin^2\theta}$ and then dividing through by $sin^2\theta$.

### Review Questions

2) You may need to remind students here not to use the `[U+0080][U+009C]`$\sin^{-1}$`[U+0080][U+009D]` function on

their calculators to find the cosecant, but instead to find the sine and then use the `[U+0080][U+009C]`$x^{-1}$`[U+0080][U+009D]` key to find the reciprocal of the sine.

3) On this and the previous problem, some students may still be confusing the domain with the range. They may also not realize that an input value that makes a function undefined is an input value that must be excluded from the function's domain.

8) Students may get the functions here mixed up with their reciprocals, or may get the Pythagorean identity backwards.

### Additional Problems

1) If $\cos\theta = \frac{24}{25}$, what is the value of $\tan\theta$?

2) If $\sin\theta = \frac{5}{13}$, what is the value of $\cot\theta$?

### Answers to Additional Problems

1) $\frac{7}{24}$. Solving this problem takes two steps: first finding $\sin\theta = \frac{7}{25}$ using the Pythagorean identity, and then finding $\frac{\sin\theta}{\cos\theta}$.

2) $\frac{12}{5}$. The problem is similar to the previous one, but students must remember that $\cot\theta$ equals $\frac{\cos\theta}{\sin\theta}$ and not $\frac{\sin\theta}{\cos\theta}$.

# Applications of Right Triangle Trigonometry

### In-Text Examples

1) Some students may have a hard time understanding which ratios to use to solve which triangles. Try to make clear to them that they should pick a ratio for which they know the angle measure and one of the two sides involved, and then use the ratio to find the other side—or pick a ratio for which they know the two sides, and use the ratio to find the angle. A trig ratio for which they only know one of those three pieces of information won't do them any good, and a ratio for which they already have all three will only tell them what they already know.

2) Using the sine to find the second leg might seem like a good idea, but it would involve plugging in the value we just found for the hypotenuse, which is an approximation. Using the tangent is better because it allows us to plug in the exact value we were given for the length of the first leg, and plugging in an exact value instead of an approximation will yield a more accurate result.

### Review Questions

1) As in example 2, students should try to use the given numbers whenever possible rather than plugging the approximate values they've found earlier into later parts of the problem. In this case, that means they should use the sine to find b and the cosine to find a, rather than finding one of them and then using the tangent or the Pythagorean Theorem to find the other.

3) Note that this problem refers to Example 2 in the earlier part of the lesson, not to either of the problems directly above.

7) Students may try to plug in $100°$ as one of the angles in the triangle, when the relevant angles are actually $80°$ and $10°$.

8) Some students may not see what this problem has to do with solving triangles; others may see that they have to divide the quadrilateral into triangles, but may pick the wrong way to do it—drawing a diagonal from upper right to lower left instead of vice versa, which makes the problem unsolvable using the tools they currently have.

9) Don't let students get hung up on trying to figure out how to find the width of the pond at its widest point; "how wide" here just means "how far is it from $A$ to $B$?"

10) Students may jump to the conclusion that $\triangle PAN$ is a right triangle and try to find $x$ based on the sine or tangent

of 50°.

**Additional Problems**

1) a) What value would you get for the height of the tree in Example 3 if you did not take the height of the person into account?

b) In example 4, why couldn't we just add 5 feet to the answer we found in the first part (where we didn't take the person's height into account) to get the answer to the second part (where we did)?

**Answers to Additional Problems**

1) a) 15.63 feet.

b) As you can see from the diagrams, taking the height of the viewer into account in example 3 just required us to shift the triangle up 5 feet. In example 4, however, accounting for the person's height required us to actually lengthen one of the sides of the triangle by 5 feet, which resulted in the proportions of the whole triangle being different. This is why drawing accurate diagrams is important in trigonometry; sometimes we need them to make it absolutely clear what information we can just assume and what information we can't.

*2.1. TRIGONOMETRY AND RIGHT ANGLES*

## 2.2 Circular Functions

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### Radian Measure

**In-Text Examples**

One very common mistake to make when working in radians is to forget from time to time that a complete rotation is $2\pi$ radians, not $\pi$ radians. This doesn't happen often when one is thinking of a whole circle, but it is very easy to think of a quarter of a circle as being $\frac{\pi}{4}$ radians instead of $\frac{\pi}{2}$, a sixth of a circle as being $\frac{\pi}{6}$ instead of $\frac{\pi}{3}$, and so on. Students may make this type of error frequently, and may continue making it for quite some time, so remind them more than once to be particularly careful about checking their angle measures when working in radians.

(As evidence that not only students are prone to this particular error, see the illustration to example 4 in the lesson immediately following this one.)

Another common error is to get the degrees-to-radians formula and the radians-to-degrees formula mixed up. The note at the bottom of page 101 should help students with this, though: basically, they should multiply by $\frac{\pi}{180}$ when they want $\pi$ in their answer (that is, when converting to radians), and should multiply by $\frac{180}{\pi}$ when they have a $\pi$ to get rid of (that is, when converting from radians to degrees). Of course, there won't always be a $\pi$ involved when working in radians (see problem 5 below, for example), but the mnemonic will remind them which way the conversion goes if they don't take it too literally.

**Review Questions**

6) This problem should read "sine" instead of "cosine"; students may therefore think that Gina's typing in "sin" instead of "cos" is the problem, instead of noticing that the calculator is in the wrong mode. They may also get $\frac{-\sqrt{3}}{2}$ instead of $\frac{-1}{2}$ as the actual value, and shouldn't be penalized for this.

**Additional Problems**

1) What is $\sqrt{2}$ radians in degrees? (Round to the nearest tenth.)

2) What is 90 radians in degrees, and what is its reference angle?

3) What is $\pi$ degrees in radians? (Round to four decimal places.)

**Answers to Additional Problems**

1) $81.03°$

2) $5156.6°$, which is coterminal with $116.6°$, so its reference angle is $63.4°$.

3) $0.0548$.

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### Applications of Radian Measure

**In-Text Examples**

1) The most common mistake to make here is to forget that the hour hand is not right on the 11, but a third of the way between 11 and 12. Students who forget this may jump to the conclusion that the hands are $\frac{5}{12}$ of the circle apart, and then may also forget (as mentioned above) that $\frac{5}{12}$ of the circle does not equal $\frac{5\pi}{12}$ radians.

2) A common error has actually been made in the text here: the numbers 12 and 11.81 have been substituted for $r$ in the arc-length formula when each of those numbers is in fact a diameter and not a radius. Students should be cautioned against making the same error on future problems, but should probably be forgiven for making it here.

4) As mentioned earlier, another common error appears in the in-text illustration here: $\frac{2\pi}{3}$ would of course be $\frac{1}{3}$ of the circle, not $\frac{2}{3}$. This will only lead to student error, though, if students try finding the area of the whole circle first and then taking $\frac{2}{3}$ of it instead of $\frac{1}{3}$ of it. If they apply the formula in the text instead, they will not go wrong, since the formula is not based on the illustration.

**Review Questions**

1) On part a.iii, students should not try to convert the approximate angle measure they found in part a.ii to degrees (as they may try to do with their calculators, especially since they have just used them to find that approximation). Instead they should start with the exact angle measure from part a.i, and convert it by hand using the formula they learned earlier.

3) Simply miscounting the dots may lead to error here. (There are 32.) Also, in finding the distance between the two dots selected in part b, students may include the dots at both the beginning and the end and conclude that the distance between them is $\frac{14}{32}$ of the circle when it is really $\frac{13}{32}$.

4) Students may forget to take into account both the radius of the outer circle and the radius of the inner circle when calculating the area of each section.

5) This problem is a particularly easy place to make the routine error of plugging in the diameter in place of the radius.

# Circular Functions of Real Numbers

**Review Questions**

1) A quick hint to use similar triangles should help students who get stuck on this problem.

3) Students may get a little confused about where to do the labeling; point out if necessary that the largest circle in the diagram is the unit circle. The other circles are just there to give them a convenient place to write the angle measures without having to write them right on top of the coordinates of the points; if they get mixed up and write the radian measures on the inner circle and the degree measures on the middle circle, there's no need to penalize them as long as they've matched the correct degree and radian measures with the correct angles.

4) Perceptive students may draw the cosine segment in a different place than the book suggests; they may draw it as a horizontal segment extending from the $y-$axis to the point where the sine segment meets the unit circle. This isn't an error; in fact, drawing the segment this way makes it easier to see that the sine and cosine segments have a relationship similar to the relationship between the tangent and cotangent or secant and cosecant segments.

(However, drawing it the standard way is fine too, and probably easier for most.)

5) The correct answer to this question is actually any combination of $\sin, \tan,$ and $\sec$.

6) Some students may get the idea that answers b and d must both be true if either one of them is true—that the tangent must get infinitely large when the cotangent gets infinitely small, because they are "opposites" in a sense. The ambiguity of the phrase "infinitely small" doesn't help; it can be taken to mean "approaching zero," but in this case it really means "approaching negative infinity." When the tangent approaches infinity, the cotangent approaches zero, not negative infinity; when the cotangent approaches negative infinity, the tangent approaches zero, and this latter case is what happens when $x$ increases from $\frac{3\pi}{2}$ to $2\pi$. (Referring to the graphs earlier in the lesson will confirm this.)

**Additional Problems**

*2.2. CIRCULAR FUNCTIONS*

1) Why does it make sense that the ranges of the secant and cosecant functions include all numbers except those between 1 and $-1$? (Hint: think in terms of sine and cosine.)

**Answers to Additional Problems**

1) The sine and cosine functions only take values between 1 and $-1$—in other words, only numbers whose absolute value is less than or equal to 1. The secant and cosecant are the reciprocals of the sine and cosine, and the reciprocal of a number whose absolute value is less than or equal to 1 will have an absolute value greater than or equal to 1. (For example, think of a fraction between 0 and 1: its numerator is less than its denominator, so its reciprocal will have a numerator greater than its denominator and so will be greater than 1.) So, the secant and cosecant functions can only take values whose absolute value is greater than or equal to 1.

# Linear and Angular Velocity

**In-Text Examples**

1) A possible overcomplication of this problem is to think that 15 feet is the "length" of the oval shape formed by the track (i.e. the major axis of an ellipse), rather than simply the track's circumference and thus the distance the car travels per circuit.

2) Students may forget to convert their final answer from miles per minute to miles per hour. If they do remember, they are likely to try to divide the miles per minute result by 60, instead of multiplying it by 60, to get the answer in miles per hour. Encourage them to slow down and think about what the answer really means: if Lois goes .047 miles in one minute, then how far will she go in 60 times one minute?

**Review Questions**

1) Careless reading may lead students to jump to the answer $\frac{7}{9}$ cm/sec. Remind them that the speed of the dial is the circumference, not the radius, divided by 9 seconds.

3) Note that Doris' horse is not 7 m from the center, but rather 7 m farther from the center than Lois', meaning it is 10 m total from the center.

4) We haven't covered scientific notation in a while, so students may be prone to make mistakes with it. Watch for answers that are off by one or two orders of magnitude.

**Additional Problems**

1) Two gears mesh with each other so that they rotate in opposite directions, with both their outer edges moving at the same linear velocity. The radius of the larger gear is 10 cm and the radius of the smaller gear is 6 cm. The smaller gear makes two revolutions per second.

a) What is the angular velocity of the smaller gear?

b) What is the linear velocity of a point on its outside edge?

c) What is the angular velocity of the larger gear?

d) How many revolutions does the larger gear make per second?

e) A peg is attached to the larger gear at a point 2 cm from its outer edge. What is the peg's linear velocity?

**Answers to Additional Problems**

1) a) $4\pi$ radians/sec, or approximately 12.57 radians/sec.

b) $24\pi$ cm/sec, or approximately 75.40 cm/sec.

c) The linear velocity of a point on its outside edge is the same as that of the smaller gear, or $24\pi$ cm/sec, so its angular velocity is $2.4\pi$ radians/sec, or approximately 7.54 radians/sec.

d) 1.2 revolutions per second.

e) The peg is 8 cm from the center of the gear, so its linear velocity is 19.2π cm/sec, or approximately 60.32 cm/sec.

## Graphing Sine and Cosine Functions

### In-Text Examples

2) Students may think the period is 2 units, because each "portion" of the graph is 2 units wide. Explain that it takes one "high" portion and one "low" portion to make up one complete cycle of the graph.

3-6) Despite the explanations in the text, it is likely that some students will still habitually mix up the period with the frequency. Repeated drilling may be the only way to fix this.

### Review Questions

2) A few students may be tripped up by part d; since there are no minimum and maximum values, they may get the idea they're supposed to be looking for something else, and supply an answer like "$\frac{\pi}{2}$ and $\frac{-\pi}{2}$" because those are the beginning and ending $x-$values of one cycle of the graph.

The error they are most likely to make on problems like $e$ and $f$ is to think that the number within the parentheses affects the maximum and minimum $y-$values. However, solving problems a through $d$ first should help reinforce that it is only the multiplier out front that matters.

3) Students may try to simply divide both sides by $\sin(x)$; they will then end up with the equation $4 = 1$, for which there is no solution. However, this technique is incorrect because $\sin(x)$ sometimes equals zero on the interval of $x-$values given for the problem, and dividing by an expression that can equal zero often eliminates possible solutions.

Graphing the two functions instead (or simply thinking about them in the right way) will show that whenever $\sin(x)$ equals zero, $4\sin(x)$ also equals zero and so the two expressions are equal. This happens at three places on the given interval, including the interval's endpoints.

A more appropriate application of algebra will yield this solution as well: instead of dividing both sides by $\sin(x)$, subtract $\sin(x)$ from both sides to yield $3\sin(x) = 0$, and then divide by 3 to get $\sin(x) = 0$.

4) Getting the period and frequency mixed up is the most likely error here; reviewing the definitions may help.

5) Students may get twice the correct value for the amplitude here, forgetting that it's the distance from the minimum or maximum to the *middle* of the graph rather than to the maximum or minimum. Also, when writing out the equation, they may again get mixed up about which number goes where.

6) At this point, students may still be stretching the graph when they should be shrinking it, or vice versa.

## Translating Sine and Cosine Functions

### In-Text Examples

1) A few students may think we're still dealing with amplitude here, and that the maximum and minimum are 6 and $-6$. Remind them that we are now shifting the graph rather than stretching it; the maximum and minimum remain the same distance apart, but have both moved together 6 units from where they started out.

### Review Questions

1-5) Starting with the questions and trying to match them to the functions right away is tempting, but almost certainly the wrong way to approach this set of problems; it's much better to sketch out graphs of the functions and then match

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them to the appropriate descriptions.

Also, the questions about $y-$intercepts may be confusing, as we haven't directly discussed those with regard to trig functions specifically. The $y-$intercepts can't be derived directly from the amplitude, frequency, period, phase shift, or vertical shift, as students may be tempted to try, but they can often be read off the graph fairly easily. The best way to find them, however, is simply to plug in 0 for $x$ and then calculate the value of $y$. (Remember, the $y-$intercept is simply the value of $y$ when the graph crosses the $y-$axis, which is to say when $x$ equals 0.)

6) "Express the equation as both a sine and cosine function" may be misconstrued to mean "write one equation that involves both sine and cosine" rather than "write one equation that involves sine and another that involves cosine."

Also, watch out for students getting the sine and cosine graphs mixed up—not just on this problem, but on any problem that involves phase-shifted sinusoids (like the next four problems!).

7-10) Students may unfortunately be thrown by the fact that graphs A and C are incorrectly drawn: they are shifted up 2 units when they should be shifted 1 unit. Explain this to avoid confusing them.

Students may also still be getting mixed up about whether a shift to the left (or to the right) should be described with a negative or positive number, and this is a particularly bad time to make that error because it may be compounded with the error of mixing up sine and cosine graphs, resulting in students mistaking a sine graph shifted one way for a cosine graph shifted the other way (or even shifted the same way).

11) It may be hard to tell here that the tick marks on the $x-$axis represent 1 unit each, rather than $\pi$ or $\frac{\pi}{2}$ units. (The multiples of $\pi$ are indicated in approximately the right positions, but without tick marks.)

## General Sinusoidal Graphs

**In-Text Examples**

1) On problems like this, students may still be getting mixed up about which transformations correspond to which parts of the equation. In particular, they may confuse the amplitude with the vertical shift and the frequency with the phase shift, or get the frequency and the period mixed up.

3) One knee-jerk error here is to assume that the amplitude is the same as the maximum value, or in this case 60. (In reality, the amplitude is indeed the same as the maximum value whenever the vertical shift is 0, but otherwise the maximum value is equal to the amplitude *plus* the vertical shift—in this case, the vertical shift is 20, so the amplitude is 40.)

Another common error is, once again, to forget that the amplitude is only half the total height of the graph.

**Review Questions**

1-5) Trying to find the maximum and minimum after finding the amplitude but before finding the vertical shift may lead students to get the wrong values, as they may default to treating the graph as if it were centered at 0.

**Additional Problems**

1) The graph of $y = 2 + \cos(3(x - \pi))$ is translated an additional $\frac{\pi}{2}$ units to the left and 3 units down. What is the equation of the new graph?

2) The graph of $y = 3 + 2\sin\left(6\left(x - \frac{\pi}{3}\right)\right)$ is stretched so that its period is twice as long.

What is the frequency of the new sine wave?

3) Which of the following yields the same graph as $y = 2 + \cos\left(2\left(x + \frac{\pi}{2}\right)\right)$?

a) $y = 2 + \cos\left(2\left(x - \frac{\pi}{2}\right)\right)$

b) $y = 2 + \sin\left(2\left(x - \frac{\pi}{2}\right)\right)$

c) $y = 2 + \sin\left(2\left(x + \frac{\pi}{2}\right)\right)$

d) $y = 2 - \cos\left(2\left(x + \frac{\pi}{2}\right)\right)$

**Answers to Additional Problems**

1) $y = -1 + \cos\left(3\left(x - \frac{\pi}{2}\right)\right)$

2) If the period is doubled, the frequency is halved. The old frequency is 6, so the new frequency is 3.

3) a

# 2.3  Trigonometric Identities

## Fundamental Identities

### Review Questions

1-3) Students may still not quite understand that they need to narrow down what quadrant the angle is in before finding the other trig functions; instead they may just assume it is in the first plausible quadrant they think of.

5) The values of sine and cosine are reversed in the answer key; the sine should be $\frac{-4}{5}$ and the cosine should be $\frac{3}{5}$.

7) Students may have a hard time figuring out where the $\theta$ they are supposed to be dealing with comes from. They may need to be walked through the first couple steps of drawing the given triangle and picking one of its angles to be $\theta$ so that they can find the sine and cosine of $\theta$ and go from there.

8) Students may not immediately see that part a is simply a difference of two squares, or they may not remember the formula for factoring a difference of squares. Thinking of $\sin\theta$ as `[U+0080][U+009C]x[U+0080][U+009D]` and $\cos\theta$ as `[U+0080][U+009C]y[U+0080][U+009D]` may make the problem look more familiar, and thinking of $\sin\theta$ as `[U+0080][U+009C]x[U+0080][U+009D]` will definitely make part b easier.

9) This is one of those fractions that will tempt students to try canceling terms that really don't cancel; they may end up thinking the whole thing can be reduced to $\sin^2\theta - \cos^2\theta$ and then trying to go from there. The actual solution, as explained in the text, is a bit tricky, as it involves factoring the numerator as a difference of squares, as they did in problem 8, but not factoring the denominator. Students who try factoring the denominator as well aren't doing anything mathematically wrong, but they will have to go through a couple of extra steps as a result.

10) Students who remember the proof in the last chapter may try to use segments on the unit circle for this proof. This isn't technically wrong, although the real point here is for them to see that they can prove the identity by expressing it in terms of sine and cosine.

### Additional Problems

1) $\sin\theta = \frac{2}{3}$ and $\tan\theta = -3$. What is the value of $\cos\theta$ and what quadrant is $\theta$ in?

2) If $\tan\theta = \frac{3}{5}$, what are the possible values of $\sec\theta$?

### Answers to Additional Problems

1) $\cos\theta = \frac{-2}{9}$ and $\theta$ is in Quadrant II.

2) The Pythagorean identity from problem 10 is the best way to solve this. $\tan^2\theta = \frac{9}{25}$, so $\tan^2\theta + 1 = \frac{34}{25} = \sec^2\theta$, so $\sec\theta = \pm\frac{\sqrt{34}}{\sqrt{25}} = \pm\frac{\sqrt{34}}{5}$.

## Verifying Identities

### Review Questions

1) Students will get stuck on this problem if it doesn't occur to them to express everything in terms of sine and cosine. You may want to stress that this is almost always a useful technique.

Also, they may get mixed up and express $\sec x$ as $\frac{1}{\sin x}$ instead of $\frac{1}{\cos x}$.

2) There is more than one approach to this problem, so it's actually hard to go wrong; the solution presented in the text is just one way of verifying the identity. However, students may have a hard time figuring out how to begin. The best thing for them to do is just think of any substitution they can usefully make, and then simplify the expression and see what seems useful to do next.

3) Students may try to simply add the denominators or otherwise go the wrong way about finding a common denominator. Also, they may (on this and the next few problems) try to substitute $\sin x$ for $1 + \cos x$ or $1 - \cos x$.

4) Cross-multiplying is the easiest way to solve this problem, but students may not think of that right away because they've been told they should usually only work on one side of the problem at a time. You may want to let them know that when both sides of the equation are fractions (each side must be a single fraction with no other terms), cross-multiplying is often a useful first step.

6) The expression $1 - 2\sin^2 b$ may look as though it can be treated like $1 - \sin^2 b$, which is equal to $\cos^2 b$. $2 - 2\sin^2 b$ would indeed be equal to $2\cos^2 b$, because of the common factor of 2, but $1 - 2\sin^2 b$ doesn't equal anything immediately useful. Simplifying the left-hand side rather than the right is the approach students should take.

7) The multiple negative signs here may lead to sign errors.

8) This is another occasion for misapplying a Pythagorean identity: students may treat $(\sec x - \tan x)^2$ as $\sec^2 x - \tan^2 x$, which equals 1.

**Additional Problems**

1) Double-check the identity from problem 1 above by verifying that it holds true for $x = \frac{\pi}{4}$.

2) Verify any other identity from the problem set above by plugging in an angle measure of your choice.

**Answers to Additional Problems**

1) Plugging in $x = \frac{\pi}{4}$ gives us $\sin\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right)$; calculating the trig values yields $\frac{\sqrt{2}}{2} \times 1 + \frac{\sqrt{2}}{2} = \sqrt{2}$, or $2 \times \frac{\sqrt{2}}{2} = \sqrt{2}$, or $\sqrt{2} = \sqrt{2}$. QED.

2) Answers will vary.

# Sum and Difference Identities for Cosine

**Review Questions**

On all the problems here, students may simply mix up the sum formula with the difference formula; the formulas are a little counterintuitive, since the sum formula involves subtracting and the difference formula involves adding.

1) Students will get stuck on this one unless they realize that $\frac{5\pi}{12}$ is the sum of $\frac{3\pi}{12}$ and $\frac{2\pi}{12}$, and that those in turn are equal to $\frac{\pi}{4}$ and $\frac{\pi}{6}$, whose trig values they are already familiar with.

2) Students may think they are done here after they find the cosines of $y$ and $z$ respectively. More commonly, they may think that they need to find the angle measures of $y$ and $z$ so they can find out what $y - z$ is in order to find its cosine—thereby missing the point of the problem, which is that they only need to find the cosines of $y$ and $z$ and then plug those, together with the sines of $y$ and $z$, into the difference formula for cosines.

3) Some students may try adding up a bunch of first-quadrant angles to get $345°$, forcing them to apply the sum formula multiple times, because they have forgotten that the trig values for key angles in the fourth quadrant are the same as those in the first.

6) Students may try canceling terms before separating the fraction into two fractions.

7) Students may forget that $\pi$ is an actual angle whose trig values they know, rather than just a variable like $\theta$. They

may get $\sin \pi$ and $\cos \pi$ mixed up, or may get them mixed up with $\sin 2\pi$ and $\cos 2\pi$ (that is, $\sin 0$ and $\cos 0$).

8) This is one problem where expressing everything in terms of sine and cosine may actually make things harder; as the solution key shows, it's easier to simply express the left side in terms of tangent.

Another way students may make the problem harder than it needs to be is by cross-multiplying. While that is often a useful technique when dealing with proportions like this, in this case it yields some very complicated expressions that take some work to simplify.

9) After applying the sum and difference formulas to the left-hand side, students are likely to get stuck. The solution key shows the most useful Pythagorean substitutions to make next, but the key insight is simply that Pythagorean identities in general are the tool to use.

10) It's tempting here to use the sum identity first, but it's a much better idea to divide by 2, take the square root of both sides, and *then* apply the sum identity. (And when taking the square root of both sides, we must remember to account for both the positive and negative solutions.)

Some students may also be tempted by the $\cos^2$ term to try using a Pythagorean identity, which will not help at all.

## Sum and Difference Identities for Sine and Tangent

**Review Questions**

1) This is another classic occasion for forgetting that $2\pi$ radians instead of $\pi$ radians make up a circle. Students who have that particular memory lapse in this case may get the idea that $\frac{17\pi}{12}$ is the same as one whole rotation plus $\frac{5\pi}{12}$—especially since the number $\frac{5\pi}{12}$ was just mentioned in the last paragraph. This won't cause their final answer to be wildly wrong—it will just be positive when it should be negative—but because the error seems so minor, they may not be able to figure out where they went wrong.

2) As on problem 3 of the previous lesson, students may try adding up a bunch of first-quadrant angles to get $345°$, forcing them to apply the sum formula multiple times, because they have forgotten that the trig values for key angles in the fourth quadrant are the same as those in the first.

3) As on problem 2 of the previous lesson, students may think they are done here after they find the sines of $y$ and $z$ respectively, or they may think that they need to find the angle measures of $y$ and $z$ so they can find out what $y + z$ is in order to find its sine, rather than simply finding the sines of the angles and then plugging them into the sum formula.

4) Some students may get stuck trying to find the sine and cosine of $5°$ and $25°$ and plug them into the given expression, which there isn't any good way to do given the knowledge they have so far. The trick here, of course, is to recognize that this problem asks them to apply a sum formula backwards; once they see that the given expression is equivalent to $\sin(5° + 25°)$, the problem becomes easy.

5) The right-hand side looks a bit like a Pythagorean identity but isn't; students should focus on simplifying the left-hand side instead of looking for ways to simplify the right.

6) As on problem 7 of the previous lesson, students may forget that $\pi$ is an actual angle whose trig values they know, rather than just a variable like $\theta$; they may also get $\tan \pi$ mixed up with $\tan 2\pi$ (that is, $\tan 0$).

7) Problems like this, with fractions in the numerator and denominator of other fractions, are a likely place for students to make basic fraction-multiplying errors. Some of them may also still have trouble rationalizing denominators.

10) The same temptation to use the sum identity first may happen here as on problem 10 of the previous lesson.

**Additional Problems**

1) Find the exact value of $\sin -15°$.

**Answers to Additional Problems**

1) Using the difference formula: $\sin(-15°) = \sin(30° - 45°)$

$$= \sin 30° \cos 45° - \cos 30° \sin 45°$$
$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{2}$$

# Double-Angle Identities

**Review Questions**

1) Students may try to find the measure of angle $x$ when they don't really need to; they'll get stuck if they try to find the trig functions of $2x$ that way instead of by using the double-angle rules.

2) Some students might get this particular double-angle formula mixed up with the Pythagorean identity and think this is a trick question whose answer is simply 1.

3) Treating the triple angle like a double angle is a trap students may fall into. They may also simply get stuck on this problem because the double-angle formulas are the only thing fresh in their minds and they've forgotten that they can also use the sum and difference formulas.

4) Expanding $\cos 2t$ and then simplifying the right-hand side is a tempting first step to try; it's not immediately obvious that this method won't get results as easily as the method outlined in the solution key will.

5) This problem is of course subject to the same error as problem 1.

6) It's very tempting here to start by moving $\sin x$ to the right-hand side of the equation. After doing that and then expanding $\sin 2x$, the next logical step seems to be to divide both sides by $\sin x$, leaving $2\cos x = -1$. This does yield two of the correct solutions to the equation, but eliminates the other two; dividing both sides by an expression that can equal zero usually eliminates solutions that shouldn't be eliminated. You may need to remind students of this fact so that they don't do something similar on other problems; in general, dividing both sides of an equation by a trig expression that can equal zero (and most of them can) is not a good idea.

8) The most likely error for students to make here is to think they are done when they get $\frac{1}{4}(\cos^2 2x + 2\cos 2x + 1)$, forgetting that the answer is supposed to be in terms of the first power of cosine (which means they need to get rid of that $\cos^2$ term). Then, they may also be unsure how to convert, or not realize they need to convert, the $\cos^2$ formula so that it works when the argument is $2x$ and not just $x$.

The same applies to problem 9.

10) This problem shouldn't be too difficult once students understand the previous problems, except that they may not think of expressing $\tan x$ as $\frac{\sin x}{\cos x}$ right away.

**Additional Problems**

1) Use double-angle identities to verify the values of $\cos \pi$ and $\sin \pi$.

**Answers to Additional Problems**

1) $\cos \pi = 2\cos^2 \frac{\pi}{2} - 1$

*2.3. TRIGONOMETRIC IDENTITIES*

$$= 2 \cdot 0^2 - 1$$
$$= 0 - 1$$
$$= -1$$

$$\sin \pi = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2}$$
$$= 2 \cdot 1 \cdot 0$$
$$= 0$$

## Half-Angle Identities

**Review Questions**

1) Students might forget that 225° is an angle they should be familiar with, although hopefully Example 1 earlier in this section will have reminded them.

In general, students may have a hard time recognizing the angles they get when they double the angles given in the first few problems as familiar ones—but since the lesson is about half-angle identities, they should at least have the general idea that doubling the angles in the problems might be useful, and then be motivated to think a little harder about whether they do in fact know the trig functions for those doubled angles.

However, they may also not realize at first that they should double the angles; it's easy to get the notions of "half angles" and "double angles" mixed up, and the fact that one has to *double* the original angle to get the right values to use in the *half*-angle formula, and vice versa, can be a bit confusing.

5) As in problem 1 from the previous lesson, students may need to be discouraged from trying to find the actual measure of angle $\theta$. They have enough information to find the value of $\cos \theta$, and they should do that and go from there. The same applies to problem 8 below.

6) The process for solving this problem involves a lot of fractions within fractions; it's quite easy to mess up here by flipping fractions that shouldn't be flipped, or multiplying fractions that should be divided, or vice versa. Some students may also still be mixing up secant and cosecant, especially when there are this many of both floating around.

Additionally, students are likely to try to solve the problem by simplifying one side only, or by simplifying just one side at a time, which has worked on many problems in the past but won't really work here. After simplifying the right-hand side a lot and the left-hand side a little, they will need to start working on both sides at once. The same applies to problem 7.

9) Students may try applying the half-angle identity before isolating $\cos \frac{x}{2}$; this isn't technically wrong, but may overcomplicate the problem and cause them to make more errors.

**Additional Problems**

1) Use half-angle identities to verify the values of $\sin \pi, \cos \pi$, and $\tan \pi$.

**Answers to Additional Problems**

1)

$$\sin \pi = \pm \sqrt{\frac{1 - \cos 2\pi}{2}}$$
$$= \pm \sqrt{\frac{1 - 1}{2}}$$
$$= \pm \sqrt{\frac{0}{2}}$$
$$= \pm \sqrt{0}$$
$$= 0$$

2)

$$\cos \pi = \pm \sqrt{\frac{\cos 2\pi + 1}{2}}$$
$$= \pm \sqrt{\frac{1 + 1}{2}}$$
$$= \pm \sqrt{\frac{2}{2}}$$
$$= \pm \sqrt{1}$$
$$= \pm 1$$
$$= -1 (\text{because cosine is negative in the second and third quadrants})$$

3)

$$\tan \pi = \pm \sqrt{\frac{1 - \cos 2\pi}{1 + \cos \pi}}$$
$$= \pm \sqrt{\frac{1 - 1}{1 + 1}}$$
$$= \pm \sqrt{\frac{0}{2}}$$
$$= \pm \sqrt{0}$$
$$= 0$$

# Product-and-Sum, Sum-and-Product and Linear Combinations of Identities

**Review Questions**

1) Some students may have trouble jumping from the abstract to the concrete in order to see how the sum-to-product rule applies here; a few may even try expressing the sum as $\sin 14x$.

6) Converting the equation to $\sin 4x = -\sin 2x$ is one tempting wrong path here, as on problem 7.

10) The sum-to-product formula won't help here.

**Additional Problems**

*2.3.  TRIGONOMETRIC IDENTITIES*

1) Express the sum as a product: $\cos 50° + \cos 30°$.

2) Verify the identity $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$ for $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{6}$.

**Answers to Additional Problems**

1)

$$\cos 50° + \cos 30°$$
$$= 2\cos\left(\frac{50° + 30°}{2}\right)\cos\left(\frac{50° - 30°}{2}\right)$$
$$= 2\cos\left(\frac{80°}{2}\right)\cos\left(\frac{20°}{2}\right)$$
$$= 2\cos(40°)\cos(10°)$$

2)

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$
$$\therefore \cos\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{1}{2}\left[\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right]$$
$$\therefore \cos\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right]$$
$$\therefore \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}\left[1 - \frac{1}{2}\right]$$
$$\therefore \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}\cdot\frac{1}{2}$$
$$\therefore \frac{1}{4} = \frac{1}{4} Q.E.D.$$

# Chapter Review Exercises

**Review Questions**

2) Some students may try to cancel terms before they factor the numerator.

4) Watch out for students multiplying fractions when they should be adding them, or vice versa.

5) Students thinking they don't know the arcsecant of 2 may try to find it with their calculators and end up with an approximation instead of an exact answer. The trick is to realize that the arcsecant of 2 is the arccosine of $\frac{1}{2}$, which they do know.

8) Some students may stop once they find $2x$, instead of going on to find $x$.

11-12) It's easy to get the angle sum identities mixed up with the sum-to-product identities; the trick is to keep straight which ones are for a single trig function applied to a sum of two angles, and which ones are for the sum of two separate trig functions. It may be useful, before assigning these exercises, to review all the identities at once and point out the similarities and differences between them.

13) This looks like a case for the sum-to-product identity at first glance, but there isn't actually a sum-to-product identity for the sum of a sine and a cosine; students might get mixed up, though, and try one of the other sum-to-product identities, ending up with a somewhat simplified but wrong answer.

14) This is a tricky problem; starting out by expressing $6x$ as $3x + 3x$ might look like the way to go, but that turns out to add several extra steps to the process, and also adds more terms that are harder to keep track of.

**Additional Problems**

1) Write as a sum: $\sin(9x) \; \cos(3x)$

2) Verify that the identity from problem 1 holds true for $x = \frac{\pi}{12}$.

**Answers to Additional Problems**

1)

$$\begin{aligned} \sin 9x \cos 3x &= \frac{1}{2}[\sin(9x + 3x) + \sin(9x - 3x)] \\ &= \frac{1}{2}[\sin(12x) + \sin(6x)] \end{aligned}$$

2)

$$\sin 9x \cos 3x = \frac{1}{2}[\sin(12x) + \sin(6x)]$$

$$\sin 9\left(\frac{\pi}{12}\right)\cos 3\left(\frac{\pi}{12}\right) = \frac{1}{2}\left[\sin\left(12\left(\frac{\pi}{12}\right)\right) + \sin\left(6\left(\frac{\pi}{12}\right)\right)\right]$$

$$\therefore \sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = \frac{1}{2}\left[\sin(\pi) + \sin\left(\frac{\pi}{2}\right)\right]$$

$$\therefore \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}[0 + 1]$$

$$\therefore \frac{2}{4} = \frac{1}{2} \cdot 1$$

$$\therefore \frac{1}{2} = \frac{1}{2} \quad Q.E.D.$$

*2.3. TRIGONOMETRIC IDENTITIES*

# 2.4 Inverse Functions and Trigonometric Equations

## Inverse Trigonometric Functions

*General Definitions of Inverse Trigonometric Functions*

**In-Text Examples**

1) Students who don't refer to the illustration may get mixed up about which angle is $\theta$. If they start with $\tan\theta = \frac{16 \text{ feet}}{10 \text{ feet}}$, they will end up with $58°$ as their answer, which is the angle the boards will make with the short side of the deck rather than the long side.

Students may also be using the wrong key on their calculators to find the inverse tangent; they may be just using the "tan" key by itself, or they may be using a combination of the "tan" key and the reciprocal $(x^{-1})$ key.

**Review Questions**

1) Some students will still be confusing the vertical and horizontal line tests here, or otherwise be fuzzy on the idea of what makes a relation a function. (Also, the inverse of relation $i$ is in fact a function.)

2) Many students will try to make side $BC$ in the diagram, rather than $AC$, equal 9 feet, thinking that the ladder is 9 feet up the wall rather than 9 feet long itself.

**Additional Problems**

1) While walking home, you decide to take a shortcut across an empty lot. From one corner of the lot, you cannot see the opposite corner, but you know the lot is 30 yards long and 20 yards wide. At what angle should you set off across the lot in order to aim directly for the opposite corner?

**Answers to Additional Problems**

1) $\tan^{-1}\left(\frac{30}{20}\right) \approx 56.31°$, so you should set off at a $56.31°$ angle from the shorter side (or a $33.69°$ angle from the longer side).

*Using the "Inverse" Notation*

**In-Text Examples**

1) Students may need a refresher on the unit circle at this point; in particular, they may have forgotten how to figure out what quadrant the reference triangle should be in.

**Review Questions**

1) This is a trick question; students may get confused by the fact that $\frac{\pi}{2}$ is a commonplace angle and may try to take the sine of it instead of the inverse sine, or they may try to take the inverse sine of it but be confused about how to find an angle whose sine is $\frac{\pi}{2}$ (which they should be, since there is no such angle).

## Exact Values of Inverse Functions

*Exact Values of Special Inverse Circular Functions*

## Review Questions

Using special triangles may be the wrong approach for these problems, as the angles here are not acute ones and therefore can't be directly found in triangles. Triangles may still be useful for figuring out the reference angles for the angles given, but that's after students figure out what quadrant the angles are in.

When using the unit circle, though, some students may need to be reminded that they can simply read off the sine and cosine values from the coordinates of the points on the circle, as the sine values is equal to the $y-$coordinate and the cosine value is the $x-$coordinate.

### *Range of the Outside Function, Domain of the Inside Function*

### In-Text Examples

1) Students may not realize yet that the easiest way to prove two functions are each other's inverses is to compose the two functions and prove that the composition always equals $x$; they may instead try to derive the one function from the other by swapping the variables, which will work but is not as easy.

### Review Questions

1) Both of these functions are actually not invertible simply because their inverses are not functions.

### *Applications, Technological Tools*

### Additional Problems

1) Find the inverse of the following function: $f(x) = 3 + 5\sin(2x - 7)$.

### Answers to Additional Problems

1)

$$\begin{aligned}
f(x) &= 3 + 5\sin(2x - 7) \\
\therefore x &= 3 + 5\sin(2y - 7) \\
\therefore x - 3 &= 5\sin(2y - 7) \\
\therefore \frac{x-3}{5} &= \sin(2y - 7) \\
\therefore \sin^{-1}\left(\frac{x-3}{5}\right) + 7 &= 2y \\
\therefore \frac{1}{2}\left(\sin^{-1}\left(\frac{x-3}{5}\right) + 7\right) &= y \\
\therefore f^{-1}(x) &= \frac{1}{2}\left(\sin^{-1}\left(\frac{x-3}{5}\right) + 7\right)
\end{aligned}$$

## Properties of Inverse Circular Functions

### *Derive Properties of Other Five Inverse Circular Functions in Terms of Arctan*

### In-Text Examples

1) Remind students to label which angle is $\theta$ when they are drawing triangles to solve these problems; otherwise they may end up taking the sine or cosine of the wrong angle.

(Of course, they can simply plug the values of $x$ into the set of equations above to solve these problems without drawing triangles. This method is faster, but may mask a lack of understanding of the actual principles involved.)

2) On part a, students may miss the part about the bottom of the screen not being directly on the ground. On part b,

*2.4. INVERSE FUNCTIONS AND TRIGONOMETRIC EQUATIONS*

they may be confused by the fact that they don't have enough information to find exact angle measures. Instead, they must find a formula for the angle measure in terms of $x$, and then figure out what value of $x$ maximizes this angle.

**Review Questions**

2) After all the previous discussion of how $f^{-1}(f(x)) = x$, students are likely to assume that $\tan^{-1}(\tan x) = x$. The only reason they're wrong is that the tangent function has to have its domain restricted in order to be invertible; inside this limited domain, $\tan^{-1}(\tan x)$ is indeed $x$, but outside of it, the values of $\tan^{-1}(\tan x)$ just repeat themselves, as students will see if they use their calculators to graph it. (Also, those vertical lines on the graph represent vertical asymptotes; that is, they are not actually places where the graph is vertical, but rather where the graph is undefined.)

**Additional Problems**

1) Express the function $2\cos(\tan^{-1} 3x)$ as an algebraic expression involving no trigonometric functions.

**Answers to Additional Problems**

1) $2\ \cos(\tan^{-1} 3x) = 2\ \cos\theta = \dfrac{2}{\sqrt{(3x)^2 + 1}} = \dfrac{2}{\sqrt{9x^2 + 1}}$

*Derive Inverse Cofunction Properties*

**Review Questions**

Remembering to get the domain restrictions right should be the only tricky part here.

***Find Exact Values of Functions of Inverse Functions Using Pythagorean Triples***

**In-Text Examples**

3) The most likely error here is getting the quadrant wrong.

# Applications of Inverse Circular Functions

***Revisiting*** $y = c + a\ \cos b(x - d)$

**In-Text Examples**

Students may make the same errors here that they made when they originally covered this topic, particularly getting the various shifts and stretches mixed up and getting the sign of the phase shift backwards.

**Additional Problems**

1) How would you express the equation from problem 1 above as a transformation of $y = \sin x$?

**Answers to Additional Problems**

1) $y = 3\ \sin(2(x + 15°)) + 2$, or $y = 3\ \sin(2(x - 165°)) + 2$

***Solving for Particular Values in Trigonometric Equations***

**Review Questions**

1) Students may still be unsure how to derive an equation from just the two points given. The trick is to realize that the horizontal distance between the two points is half the period (and the frequency is $2\pi$ divided by the period); the vertical distance between them is the amplitude; the average of the two heights is the vertical shift; and the horizontal distance between the maximum point and zero (divided by $2\pi$) is the phase shift (if we are using a cosine function to model the graph).

**Additional Problems**

1) You are riding a Ferris wheel at an amusement park. 15 seconds after the wheel starts turning, you are at the top of the wheel, 100 feet high. 20 seconds later, you are at the bottom of the wheel, 10 feet off the ground. At what

time did you first reach a height of 70 feet?

**Answers to Additional Problems**

1) The equation that models this problem is $55 + 45 \, \cos \frac{\pi}{20} \left( x - \frac{15}{2\pi} \right)$. Solving for $x$ in terms of $y$ gives us:

$$x = \frac{\cos^{-1}\left[\frac{y-c}{a}\right]}{b} + d \quad y = 70, \ c = 55, \ a = 45, \ b = \frac{\pi}{20}, \ d = \frac{15}{2\pi}$$

$$\therefore x = \frac{\cos^{-1}\left[\frac{70-55}{45}\right]}{\frac{x}{20}} + \frac{15}{2\pi}$$

Using a calculator gives us an answer of approximately 10.22 seconds after the Ferris wheel began moving.

# Trigonometric Equations

## *Solving Trigonometric Equations Analytically*

### **In-Text Examples**

1) Students may get stuck on these if they don't remember the basic identities.

2) Make sure to point out on part b that there are a total of four solutions on the given interval.

### **Review Questions**

1) Of course, students are likely to miss the fact that they need to double the interval of possible solutions (as explained in the solution key). Even if they do figure this out, they may not realize that this means the two solutions within the interval $[0, 2\pi)$ must each have $2\pi$ added to them to make the other two solutions. And finally, they may forget to divide the solutions by 2 at the end to get the values for $\theta$ rather than $2\theta$.

2) As on other problems, students may forget to look for solutions in more than one quadrant. They may also forget to consider the cosines of both $\frac{1}{4}$ and $-\frac{1}{4}$.

3) The trick here is to treat $4\theta$ as a double angle and $2\theta$ as the corresponding single angle; students may not think of this right away. Also, they may make the same mistakes as in problem 1.

4) The same errors as on problem 1 apply.

## *Solve Trig Equations (Factoring)*

### **In-Text Examples**

2) Students will of course be tempted to divide both sides of the equation by $\tan x + 1$; the only problem with this is that it will eliminate possible solutions because $\tan x + 1$ can equal zero. The way to get around this is to find the solutions for $2 \, \sin x = 1$, but then also find the solutions for $\tan x + 1 = 0$. The solution presented in the text is another way of doing this.

### **Review Questions**

1) Upon finding that $\sin x$ can equal either 3 or $-1$, students may be stumped by the 3, since that isn't a possible value for $\sin x$. It doesn't mean they've done anything wrong, it just means they should discard that solution.

2) Again, beware of dividing through by $\tan x$, as this will eliminate the solution $x = 0$.

## *Solve Equations (Using Identities)*

### **Review Questions**

1) Getting the signs reversed when factoring is the most likely error here.

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2) Students are likely to forget the "for all values of $x$" part; they may also forget the principal solution $\frac{3\pi}{2}$.

# Trigonometric Equations with Multiple Angles

*Solve Equations (with Double Angles)*

**In-Text Examples**

1) Again, beware of dividing both sides by $\cos x$.

4) ...or by $\sin x$.

**Review Questions**

1) Finding the actual measure of angle $x$ is tempting but not necessary.

2) The solution to part b involves realizing that $\cos^4 \theta - \sin^4 \theta$ can be treated as a difference of squares, which may escape students at first.

3) This equation is actually true for all values of $\theta$.

*Solving Trigonometric Equations Using Half Angle Formulas*

**In-Text Examples**

3) Students may be momentarily stumped by the appearance of the unfamiliar-looking angle $\frac{7\pi}{6}$. Rewriting it as $\pi + \frac{\pi}{6}$ may help them see that it is simply a third-quadrant angle with a reference angle of $\frac{\pi}{6}$, whose sine and cosine they should be able to figure out easily.

**Review Questions**

3) Students will get three solutions to this equation if they follow the procedure outlined in the solution key, but not all of those solutions will in fact be correct. Squaring both sides of an equation introduces extraneous solutions, so they must check their answers afterward to see which ones need to be discarded. This is true any time both sides of an algebraic equation are squared.

*Solving Trigonometric Equations with Multiple Angles*

**In-Text Examples**

2) Once again, remind students that they must look for values of $2\theta$ between 0 and $4\pi$ in order to find values of $\theta$ between 0 and $2\pi$.

**Review Questions**

2) Students may get stuck trying to figure out how to express $\cos 3x$ in terms of $\cos x$, perhaps by trying to refer back to the "triple-angle formula" derived in chapter 3, or trying to derive one themselves from the double- and half-angle formulas they already know. This isn't necessary, though; they only need to solve for $\cos 3x$ and then find possible values for $3x$ that lie between 0 and $6\pi$ in order to find the possible values of $x$ that lie between 0 and $2\pi$.

3) After finding the principal solutions to the equation, students may try to derive the other solutions by adding $2\pi k$, rather than $\pi k$. (If they were dealing with the sine or cosine functions instead of the tangent function, they'd be right, but the tangent function repeats its values every $\pi$ rather than $2\pi$ units.)

# Equations with Inverse Circular Functions

*Solving Trigonometric Equations Using Inverse Notation*

**In-Text Examples**

2) Students may be momentarily confused by the lack of "arc"-function buttons on their calculators, and may need to be reminded to use the `[U+0080][U+009C]` $\sin^{-1}$, `[U+0080][U+009D][U+0080][U+009C]` $\cos^{-1}$, `[U+0080][U+009D]`, and `[U+0080][U+009C]` $\tan^{-1}$ `[U+0080][U+009D]` functions instead.

**Review Questions**

3-4) Students are likely to stop when they have found the one solution within the restricted range, and forget that they are looking for solutions between 0 and $2\pi$ rather than just between 0 and $\pi$.

*Solving Trigonometric Equations Using Inverse Functions*

**In-Text Examples**

3) The most likely error here is for students to try "unwrapping" the right-hand side of the equation in the wrong order, for example by taking the inverse cosine before dividing by $A$, or by subtracting $\phi$ before it is properly isolated. This sort of error is often more likely when working with expressions that have a large number of variables, as students can get a little bit lost when they are not working with concrete numbers.

**Review Questions**

1) This problem is subject to the same error as example 3 above; also, students who do not recognize that the problem relies on an angle sum identity will approach it wrong from the beginning and get an answer that makes no sense, if they get any answer at all.

*Solving Inverse Equations Using Trigonometric Identities*

**In-Text Examples**

1) This example is much trickier than it seems at first, because of the double angle. Watch out for students forgetting to account for the 2 and just solving for $\cos(\sin^{-1} x)$, which is $\sqrt{1-x^2}$.

**Review Questions**

1) Students may get their double- and half-angle identities mixed up here, or may forget that if they are applying a double-angle identity to $\sin 2\theta \cos 2\theta$, they will end up with $\sin 4\theta$ rather than $\sin 2\theta$.

2) (Note: 10 feet is actually the distance from one end to the other, rather than the width of the one end.)

Students may think they don't have enough information to solve this problem, but they do if they express the legs of the triangles as $\sin \theta$ and $\cos \theta$; then they can set up an equation for the area of one end (and thus the volume) in terms of $\theta$. (They may need calculators, though, to find the equation's maximum value.)

# 2.5 Triangles and Vectors

## The Law of Cosines

**In-Text Examples**

1) A common error when working with the Law of Cosines is to forget to subtract, rather than add, the last term ($2ab\cos C$). Also, since students will be using their calculators for many of these problems, they may need to be reminded to check that they are in the right mode. (Degrees, rather than radians, will almost always be the mode used for problems dealing with angles in triangles.)

2) In cases where students know all three side lengths of a triangle and are trying to find one of the angles, they may fail to plug the side lengths into the formula in the correct places. A mnemonic that may help is "the length of the side *opposite* the angle we are trying to find goes on the *opposite* side of the equation from the other two side lengths."

**Review Questions**

4) Students may forget here that $23.3°$ is the measure of $\angle ABD$ rather than $\angle ABC$.

6) Once students have found the correct measure of the angle or side, they may well forget to subtract it from the given measure to tell how far off the given measure is.

10) Part a should read "If the tee is 329 yards…." as in the diagram.

On part b, students may be a little confused about how to set up the new diagram. Basically, it should look like the previous diagram, except with [U+0080][U+009C]98 yds[U+0080][U+009D] in place of [U+0080][U+009C]235 yds[U+0080][U+009D] and [U+0080][U+009C]3°[U+0080][U+009D] in place of [U+0080][U+009C]9°.[U+0080][U+009D]

12) The instruction "use some right triangle trig" may give some students the idea that the triangle described in the problem is a right triangle. Other students may need to be reminded of the formula for the area of a triangle. Others may overcomplicate the problem by trying to find the measures of more than one angle of the triangle when just one will do.

14) Some students may try to treat this quadrilateral as a parallelogram and find the area based on that formula, instead of finding the areas of the two triangular subsections separately.

15) Students are liable to think that $17°$ or $82°$ is the interior angle of the triangle that lies between the two sides of 20m and 4m length, when it is really the exterior angle (and thus the supplement of the interior angle).

More rarely, students may think that $17°$ is the measure of one angle of the triangle and $82°$ is the measure of another of the angles, instead of using $17°$ as an angle measure in part a and $82°$ as the measure of the same angle in part b.

16) Since the length of $AB$ is not given, students may try to subtract 5 cm from one of the three other lengths that are given in the diagram. Impress upon them that this is a two-part problem: first they need to find the length of $AB$ when $\angle AEH$ is $120°$, and then they must subtract 5 cm from that length and apply the Law of Cosines in reverse to find the new measure of $\angle AEH$.

## Area of a Triangle

**In-Text Examples**

2) It's very easy, when using Heron's Formula, to slip up and forget to divide by 2 when calculating $s$. It's also easy to overlook the factor $s$ under the square root sign, as it is somewhat dwarfed visually by the other three factors.

**Review Questions**

1) In parts $b$ through $d$, students may think they are supposed to find the areas of the right triangles drawn in with dotted lines, instead of the oblique triangles drawn with solid lines—or they may think, because of the right triangles, that they should use the [U+0080][U+009C]$\frac{1}{2}bh$[U+0080][U+009D] formula every time.

Also, they may try to actually find the areas instead of merely stating which formula to use to find them, but that isn't so bad in this case since it's what the next question asks them to do anyway. This also applies to problems 3 and 4.

2) Rounding off too early may get some students in trouble here; remind them if necessary not to round off until the very last step so that roundoff errors don't accumulate.

5) On this problem, students may be tempted to plug in 375 feet as the height of each triangle in the formula $\frac{1}{2}bh$, instead of realizing that 375 feet is the length of the diagonal legs of each triangle and that they must use Heron's formula to calculate the areas. Also, they may forget to quadruple the area they find for one triangle to get the total area for all four.

6) Students may need to be reminded here that the contractor will have to buy a whole number of bundles.

7) 14955.6 square yards is the new area, but the question asks for the difference between the new area and the old.

9) After plugging the area of the triangle into the $\frac{1}{2}bh$ formula, students will find that the length of the triangle's base is approximately 39.34 units; they may think this is their final answer and fail to subtract 14.4 from it to get 24.94.

**Additional Problems**

1) Use the identity from problem 10 to prove the Pythagorean Theorem; that is, show that $f^2 = d^2 + e^2$ for any right triangle $DEF$ where $f$ is the hypotenuse.

**Answers to Additional Problems**

1) Start with the identity from problem 10:

$$d^2 + e^2 + f^2 = 2(ef\cos D + df\cos E + de\cos F)$$

Now if $F$ is a right angle, $\cos F = 0$, so that term drops out and we have:

$$d^2 + e^2 + f^2 = 2(ef\cos D + df\cos E)$$

Drawing the triangle will show us that $\cos D = \frac{e}{f}$ and $\cos E = \frac{d}{f}$, so we can substitute:

$$d^2 + e^2 + f^2 = 2\left(ef \cdot \frac{e}{f} + df \cdot \frac{d}{f}\right)$$

Canceling out the $f$s:

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$$d^2 + e^2 + f^2 = 2(e^2 + d^2) = 2(d^2 + e^2)$$

And subtracting $(d^2 + e^2)$ from both sides:

$$f^2 = d^2 + e^2$$

## The Law of Sines

**Review Questions**

1) Part c is a trick question; it is neither AAS nor ASA but AAA, which we haven't covered yet.

2) Because of problem 1c, students may assert that the triangles in the chart don't all have anything in common, or don't have anything in common except that we know at least two angles. This is technically correct, but the problem is really trying to get at what the ASA and AAS cases have in common, ignoring the AAA case.

Students also shouldn't be penalized for answering that in both cases we know two angles (or can find three angles) and one side.

3) Again, part c can't be solved, so don't let students get stuck on it. (You may want to stress the "if possible" part of the instructions before they get started.)

6) Students may mix up the ASA and SAS cases here, leading them to mix up the Law of Sines and Law of Cosines cases in turn. Also, they may try to actually solve the triangles instead of just stating how they would solve them.

7) The $x$ we are solving for here is the "other half" of the base of the triangle.

8) After doing all the steps needed to find the various distances, students may forget to subtract the old distance from the new distance, or may forget to calculate the time the extra distance took as their final step.

9) Students may forget to have the driver start at the warehouse rather than at stop $A$, or may forget that she needs to get back to the warehouse after stop $C$; in other words, they may forget to add in that extra 1.1 miles of distance either or both times.

Also, they may label the angles wrong, for example labeling the $103°$ angle at stop $B$ as the exterior angle rather than the interior angle of the triangle; this is an easy mistake to make here, since they may unconsciously be thinking of the angles the two other streets make with First Street as both being in "standard position" because First Street is horizontal.

Finally, they may forget to add in the 2 minutes for each package when calculating the time, or may add 2 minutes just once instead of three times. Or, they may get the time constraints backwards, thinking of the driver as leaving at 10:00 and trying to calculate when she will get back, instead of calculating when she must leave to get back by 10:00.

10) There is more than one way to set up this problem, depending on how one interprets the descriptions given. Students therefore may not get the answer given, but should not be penalized if they can show that they have set up the problem in a way that seems reasonable.
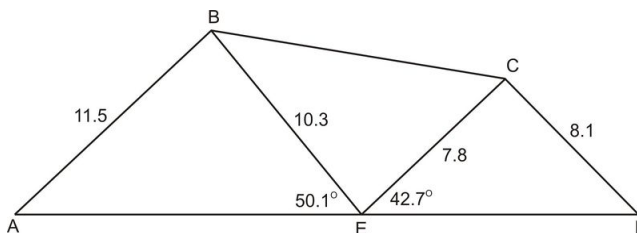
## The Ambiguous Case

**Review Questions**

3) Students may approach this problem in reverse, by rewriting the given equation as $\frac{a}{c} - \frac{c}{c} = \frac{\sin A}{\sin C} - \frac{\sin C}{\sin C}$ and going from there, ending up with the Law of Sines as their final step. This is an equally valid solution.

4) Students may have a hard time coming up with an appropriate set of sides and angle if they try picking the sides first; they will have much better luck if they think of starting with an angle $A$ and a side $b$ and then finding a side $a$ that is less than $b$ and greater than $b \sin A$. Even then, though, this will only work if they pick an acute $A$, although it may not be immediately obvious why. (Drawing a picture may make it clearer: if $A$ is the biggest angle, then $a$ must be the biggest side and certainly can't be smaller than $b$.

5) Students may think there is only one value of $A$ in each case, rather than a whole range of values.

6) Solving this problem without using the Law of Sines at all is actually possible, and students should probably not be penalized for doing so.

7) Students may get stuck if they try to find all the sides and angles in the order specified; they should be encouraged to find them in whatever order they can. Also, make sure they realize that it's $\angle ABC$ that measures $109.6°$, rather than any of the other angles whose vertices are at $B$.

8) Some students may jump to the conclusion that the triangle shown is a right triangle, simply because it looks like one.

10) The last question can't actually be answered using the information given.

**Additional Problems**

1) In the figure below, $AB = 11.5, BE = 10.3, EC = 7.8, CD = 8.1, \angle AEB = 50.1°$, and $\angle CED = 42.7°$.



Find the following, to the nearest tenth of a unit:

a) $\angle BEC$

b) $BC$

c) $\angle EBC$

d) $\angle ECB$

e) $\angle BAE$

f) $\angle ABE$

g) $AE$

h) $\angle EDC$

i) $\angle ECD$

j) $ED$

**Answers to Additional Problems**

1) a) $87.2°$ (because $\angle AED$ is a straight angle)

b) 12.6 (by Law of Cosines)

c) $38.2°$ (by Law of Sines)

d) $54.7°$ (by Law of Sines)

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e) 43.4° (by Law of Sines)

f) 86.5° (by Triangle Sum Theorem)

g) 15.0 (by Law of Sines)

h) 40.8° (by Law of Sines)

i) 96.5° (by Triangle Sum Theorem)

j) 11.9 (by Law of Sines)

## General Solutions of Triangles

**Review Questions**

3) On part d, students may think that, as on part e, we are still missing side $c$ and angle $C$. However, the fact that there is no solution possible to this triangle actually means that there is no such triangle, so technically we are not "missing" that last side and angle measure because the side and angle themselves do not exist.

5) Students may need to be reminded that a rhombus has all four sides the same and that both pairs of opposite angles are congruent. They may also need to be reminded that they can use Heron's formula to find the area of half of the rhombus, and they may forget to double that area to find that of the whole rhombus.

6) The likeliest error here, once students realize they need to divide the pentagon into triangles to get anywhere with it at all, is for them to divide it into triangles they can't solve. This will happen if they connect the vertices whose angles are already known, dividing them up into unknown angles. Instead, they need to connect the three vertices whose angles aren't given; then they will have triangles they can solve.

7) There isn't enough information given to solve the triangle; you may need to supply a couple of angle or side measures from the answer key so that students can find the rest.

**Additional Problems**

1) Find the area of the quadrilateral below, to the nearest hundredth.
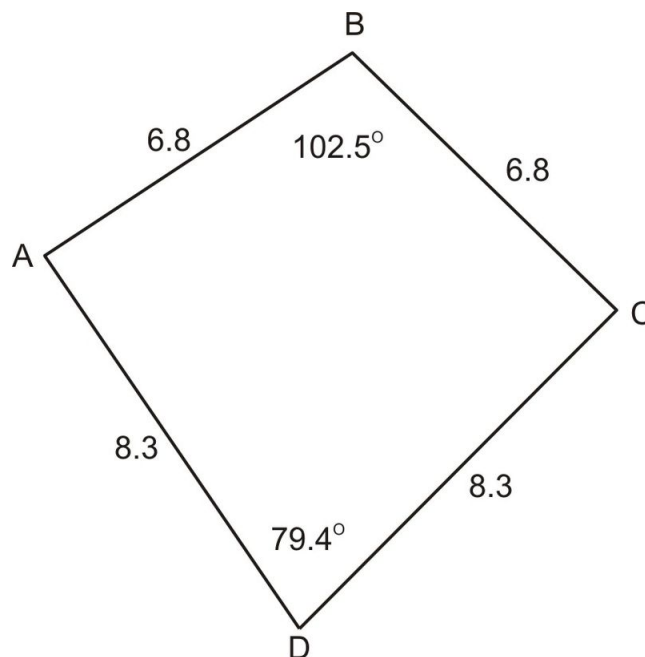


2) Find the missing angle measures in the above quadrilateral, to the nearest tenth.

**Answers to Additional Problems**

1) First, divide the quadrilateral into two triangles by drawing diagonal $AC$. Then the area of each triangle can be found with $K = \frac{1}{2}bc\sin A$.

The area of $\triangle ABC$ is 22.57, and the area of $\triangle ADC$ is 33.85. The total area of the quadrilateral is therefore 56.42.

2) The fact that $ABCD$ is symmetrical is what makes this problem possible to solve. Divide $ABCD$ into two triangles by drawing diagonal $BD$; these two triangles are congruent by SSS, and so each of them has area 28.21 (half of the area of $ABCD$). The area formula $K = \frac{1}{2}bc\sin A$ can now be used to find the missing angle in each triangle.

Both missing angles, $\angle A$ and $\angle C$, measure $88.5°$.

# Vectors

### In-Text Examples

1) Stress that the distance formula is just a version of the Pythagorean Theorem: the magnitude of the vector can be thought of as $\sqrt{(\text{difference between x} - \text{coordinates})^2 + (\text{difference between y} - \text{coordinates})^2}$.

### Review Questions

1) The answers given in the text include the vector directions with respect to $\vec{m}$—that is, the direction of each resultant vector is expressed as the angle the vector makes with $\vec{m}$—so students will get the same answers as those in the text if they too express their resultant vectors in terms of $\vec{m}$. If you don't specifically tell them to do this, though, they may express them in terms of $\vec{n}$ instead, in which case the angles they give as their answers should be the complements of the angles given in the book. (Either way is correct in the absence of more specific instructions.)

2) Students can use either the triangle or parallelogram method here. Since they will need to use a ruler and protractor, make sure their copy of the text is the right size or they will get the wrong answers. (It may be worth double-checking by measuring the lengths of the vectors $\vec{a}$ through $\vec{d}$ as they appear on the page.)

Also, note that the diagrams on the first two lines of the chart are not to scale; students will have to re-draw the vectors to get the correct magnitude and direction.

And finally, they will be measuring angles from the horizontal when there isn't a horizontal line drawn in the text, so they will have to estimate them as best they can; don't be too strict about their getting the angles exactly right.

3) This is actually only true if $\vec{a}$ and $\vec{b}$ have the same or opposite direction.

4) The direction angle, $4.6°$ NW, can also be expressed as an angle of $94.6°$ in standard position.

8-10) Students will have to draw some fairly large vectors to solve these problems, and they will still need to scale them down in order for them to fit on one page. As a result, they may not be able to get their answers accurate to the nearest tenth or even the nearest whole unit. (They won't have this problem if they use the Law of Cosines and the Law of Sines instead of drawing the vectors, but since that technique hasn't been covered yet, it probably won't occur to them to use it and they certainly shouldn't be expected to.)

Also, on problems 8 and 10, note that the angle between the two vectors in each case is the angle between them when they are placed tail-to-tail. This makes the parallelogram method fairly easy to apply, but students who use the triangle method instead may think that the angle given is the angle between the vectors when they are placed tip-to-tip, and will get the wrong answer as a result.

*2.5. TRIANGLES AND VECTORS*

## Component Vectors

**In-Text Examples**

5) Simply multiplying the coordinates by 2.5 without translating first yields a directed segment from $(10, 17.5)$ to $(30, 27.5)$. This answer isn't wrong, as it has the same magnitude and direction as the one from $(4, 7)$ to $(24, 17)$, but you may want to make sure students understand that the answers are equivalent.

**Review Questions**

2) Students may try to solve these problems by translating the vectors to the origin and reading off the coordinates of the new terminal point, or they may try to solve them by finding the difference between the $x-$coordinates of the two points and the difference between the $y-$coordinates of the two points. It may be worth pointing out that these two methods are in fact mathematically equivalent; both involve performing essentially the same operations and both yield the same answer.

8) This question may confuse students a bit; some of them may try to solve it as if they were given two components and asked for the resultant vector, while others may try to solve it as if they were given the resultant and one component and asked for the other component. Since the text isn't entirely clear on which one is actually being asked for, it's best to accept any answer that a student can justify.

9) This question is similarly ambiguous; as a result, students may get $11.5°$ rather than $11.3°$ as their answer, and this should be accepted.

10) You may need to clarify the notation here: $AB$ means simply the directed line segment from point $A$ to point $B$.

**Additional Problems**

1) Find the single ordered pair that represents $\vec{a}$ in each equation if you are given $\vec{b} = (1, 2)$ to $(3, 8)$ and $\vec{c} = (3, 3)$ to $(5, 2)$.

a) $\vec{a} = \frac{1}{2}\vec{b}$

b) $\vec{a} = -5\vec{c}$

c) $\vec{a} = \vec{b} - \vec{c}$

d) $\vec{a} = 3\vec{c} + 2\vec{b}$

**Answers to Additional Problems**

1) (By translating $\vec{b}$ and $\vec{c}$ to the origin, you can represent each of them as a single ordered pair, which makes it easier to represent $\vec{a}$ as a single ordered pair.)

a) $(1, 3)$

b) $(-10, 5)$

c) $(0, 7)$

d) $(10, 9)$

## Real-World Triangle Problem Solving

**Review Questions**

2) To solve this problem, we need to assume that the height of the canyon wall on the hiker's side is the same height as on the opposite side. (Most students will assume this anyway, but those who don't should be told to.)

5) Some few non-pool-playing students may need to be told which ball is which in the diagram, and may need to be

told that the pockets are at the corners of the table.

7) Some students may not immediately see that they need to find the area of the triangle between the three docks. Others may find the area and forget to multiply by $5.2 \times 10^{13}$ to find the number of bacteria.

8) Students may jump to the conclusion that $37°$ is the measure of the angle from tower $B$ to tower $A$ to the fire (it's actually the angle's complement.) They may also jump to the conclusion that the angle from tower $A$ to tower $B$ to the fire is a right angle.

10) Students may think the angle between the first part of the trip and the second part of the trip is $34°$. Instead, they need to think in terms of vectors; they should draw the two halves of the trip as individual vectors with their separate headings, and then find their resultant.

### Additional Problems

1) a) A ship approaches close enough to shore to spot a famous lighthouse which is known to be 180 feet tall, and which stands at the top of a $2200-$foot cliff. The captain, looking from a point 15 feet above the water, observes that the angle of elevation to the top of the lighthouse is $4.74°$. How far is the ship from the lighthouse, to the nearest foot? To the nearest tenth of a mile? $(5280 \text{ feet} = 1 \text{ mile}.)$

b) The lighthouse keeper notices that the ship appears to be headed toward a dangerous reef. The reef is known to be 2.8 miles from the lighthouse, and the lighthouse at this moment makes an angle of $30.8°$ with the ship and the reef. How close is the ship to the reef?
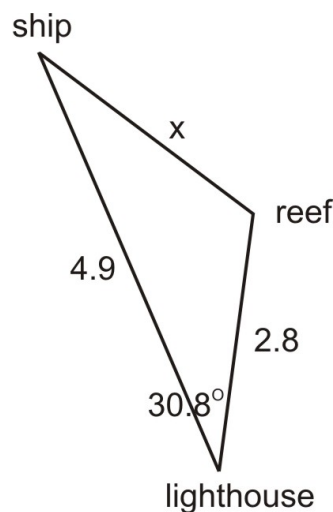
### Answers to Additional Problems

1) a)
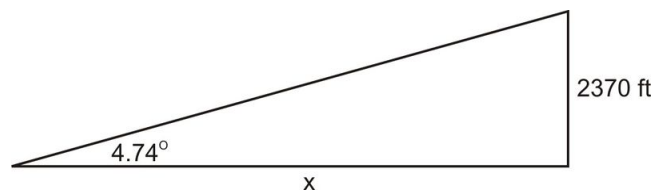


$$\tan 4.74° = \frac{2370}{x} \quad x = \frac{2370}{\tan 4.74°} \quad x = 28,522$$

The ship is $28,522$ feet, or 4.9 miles, from the lighthouse.

b)

Using the Law of Cosines:

$$x^2 = (4.9)^2 + (2.8)^2 - (2 \times 4.9 \times 2.8 \cos(30.8°))$$
$$\therefore x \approx 2.9$$

The ship is 2.9 miles from the reef.

# 2.6 Polar Equations and Complex Numbers

## Polar Coordinates

*Polar Coordinates*

**In-Text Examples**

2) You may want to stress that if $(4, 120°)$ is one way of representing the given point, $(-4, -120°)$ is *not* another way. In general, if we change the sign of the $r-$coordinate, we cannot simply change the sign of the $\theta-$coordinate to keep the point the same.

(Here's part of the reason why: When we change the sign of the $r-$coordinate, we're actually rotating the point $180°$ about the origin—or reflecting it across both the $x-$ and $y-$axes, which is the same thing. When we change the sign of the $\theta-$coordinate, we're reflecting the point across the $x-$axis. So if we change the signs of both coordinates, those two transformations combined add up to one reflection across the $y-$axis. Explaining this is optional, though; the important part is that changing the signs of both coordinates doesn't get us back to the same point.)

**Review Questions**

2) Students' answers should include the original coordinates $\left(-4, \frac{\pi}{4}\right)$ rather than the third pair of coordinates given in the answer key.

*Sinusoids of One Revolution*

**In-Text Examples**

5) Students may have trouble graphing looped limaçons like this if they don't include enough values in their table. Also, it may be hard for them to keep track of which order the points should be connected in, especially when the $r-$values become negative.

**Review Questions**

1) All the curves here are limaçons, but that answer is not sufficient.

2) The rose will have $n$ petals if $n$ is odd and $2n$ petals if $n$ is even, but that may not be apparent just from the two examples given. Students shouldn't get hung up on trying to find a more specific relationship than the one given in the answer key.

*Applications, Trigonometric Tools*

**In-Text Examples**

2) Students may think that the total distance $\theta$ they must plug into the area equation is $\frac{\pi}{3}$ rather than $\frac{2\pi}{3}$, because of the $\frac{\pi}{3}$ coordinate used earlier. The graph should help them avoid this mistake, though.

3) Any three roses will do, not just the three shown. Creating a quilt can't easily be done on just one set of axes, though; it will need to be done by copying one graph several times onto a sheet of paper.

# Polar-Cartesian Transformations

*Polar to Rectangular*

**Review Questions**

1) It's common to get the equations $x = r\cos\theta$ and $y = r\sin\theta$ backwards when solving a problem, and think that $y = r\cos\theta$ and $x = r\sin\theta$; also, it's common to think of $\tan\theta$ as being equal to $\frac{x}{y}$ instead of $\frac{y}{x}$. Watch for these errors on all problems of this type.

Also, if students use calculators on this problem they may forget to switch modes on part B.

*Rectangular to Polar*

**In-Text Examples**

1) You may need to stress the part about adding $\pi$ to the arctangent when $x$ is less than 0, so that students remember it when solving future problems.

**Review Questions**

1) Students working in degrees instead of radians will get $A(5.39, 111.73°)$ and $B(6.40, -38.39°)$ as their answers.

*Conic Section Transformations*

**In-Text Examples**

1) Mixing up the $x$`[U+0080][U+0099]`s and $y$`[U+0080][U+0099]`s is a common error when dealing with conics. When working with parabolas in particular, students may default to treating every one (if they don't sketch it out carefully first) as if it opened up or down instead of left or right.

5) You may want to point out that plugging in a negative value for $a$ doesn't yield a circle with a negative radius (what would that even look like?), but rather yields a circle of radius $a$ that's on the "other" side of the origin—left instead of right for a cosine equation, and down instead of up for a sine equation. (This makes sense if you picture what happens to a graph like this as a decreases toward zero and finally passes zero—the circle shrinks closer and closer to the origin until it passes through the origin and comes out the other side. It also makes sense if you remember that multiplying $r-$values by $-1$ is the same as rotating the graph $180°$.)

**Review Questions**

1) Students may try graphing the equation to prove it is a parabola, but this is more time-consuming and prone to error than the method described in the answer key.

2) Again, watch for students mixing up $x$ and $y$, or $a$ and $b$, or $h$ and $k$.

*Applications, Technological Tools*

**Rectangular Form or Polar Form**

3) You may need to explain that the sun is at one focus of the ellipse, so the perihelion is the distance from $F$ to $P$ and the aphelion is the length of the major axis minus the perihelion.

# Systems of Polar Equations

*Graph and Calculate Intersections of Polar Curves*

**In-Text Examples**

3) Students may be very confused by the idea that $(0,0)$ and $\left(0, \frac{\pi}{2}\right)$ represent the same point—it makes sense that adding $2\pi$ to the $\theta-$coordinate of a point yields another name for the same point, but it doesn't make sense that

adding $\frac{\pi}{2}$ would do so. In fact, the only reason it works in this case is that the $r-$coordinate of the point is 0. If we rotate through any angle whatsoever, but then go 0 units from the origin, we end up at the origin—so any pair of coordinates where the $r-$coordinate is 0 is just one more way of naming the point $(0,0)$, no matter what the $\theta-$coordinate is.

4) It may seem a little strange that the polar and rectangular coordinates for the two points of intersection are exactly the same. However, this is generally true for any point whose $\theta-$coordinate is 0: the $y-$coordinate in rectangular form will also be 0, and the $x-$coordinate will be the same as the $r-$coordinate. Doing the conversion algebraically will confirm this, and it can be handy to know.

**Review Questions**

1) The graph of these equations is deceptive; it certainly looks as though they intersect three times rather than just one. But tracing the graph of $r = 3\ \sin\theta$ will show that at two of the apparent intersection points, the $r-$value is actually negative and so is not the same as the $r-$value of $\sin 3\theta$ at the same point. Solving the system algebraically will also show that there is really only one solution.

*Equivalent Polar Curves*

**In-Text Examples**

1) Part b should read $5\cos(-90)$ rather than 2; also, the number 90 may make it seem like we are working in degrees here, but we're actually still working in radians. (The equations are basically equivalent to $r = 2.24$ and $r = -2.24$ respectively. If we were working in degrees, those same equations would be equivalent to $r = 5$ and $r = -5$.)

In general, the graph of $r = a$ is equivalent to the graph of $r = -a$.

**Review Questions**

2) There's a small chance students will get mixed up here and substitute $\theta$ for $\pi$ in their calculations, because they are used to $r$ being expressed in terms of $\theta$. In fact, the equations are simpler than they look; since $\frac{\pi}{3}$ is a constant, the whole right-hand side of each equation is a constant (it works out to simply 5.5), so the graph is just a circle centered at the origin.

*Applications, Technological Tools*

**In-Text Examples**

You might want to point out in the example here that we are concerned with both the "real" points of intersection and the "apparent" ones, since this is a real-world problem where we are concerned with the shape of the graph and not just the actual number values of the points.

# Imaginary and Complex Numbers

*Recognize*

**In-Text Examples**

A very common error when simplifying square roots is to pull numbers or expressions out from under the radical sign without actually taking their square roots; this tends to happen when there is more than one expression under the radical sign to deal with. For instance, in example 1a, students might express the answer as $16i$ rather than $4i$; or in example 2c, they might express the answer as $ix$ rather than $i\sqrt{x}$.

**Review Questions**

1) On parts c and d, students might make the error they were warned against in the text and come up with positive 15 and 35 as answers.

*Standard Form of Complex Numbers $(a+bi)$*

**In-Text Examples**

2) Students are likely not to fully grasp that they can treat the real terms and the imaginary terms completely separately, and that this equation therefore can really be thought of as two separate equations, one which can be solved for $x$ and the other for $y$. They may also forget momentarily that $i$ is not a variable, and start vaguely trying to solve for it as well as for $x$ and $y$, or may just get the idea that there is one equation here with three variables and hence not enough information to find a solution.

**Additional Problems**

1) Find the conjugate of each complex number:

a) $3 + 0i$

b) $10$

**Answers to Additional Problems**

1) a) $3$ [U+0080][U+0093] $0i$ or simply $3$

b) $10$ [U+0080][U+0093] $0i$ or simply $10$

*The Set of Complex Numbers*

**Additional Problems**

1) What sets does each number belong to?

a) $\sqrt{25}$

b) $\frac{\pi}{3}$

c) $\sin \frac{\pi}{3}$

d) $\cos \frac{\pi}{4}$

**Answers to Additional Problems**

1) a) complex, real, rational, integer, whole number, natural number

b) complex, real

c) complex, real, rational, fraction

d) complex, real

*Complex Number Plane*

**Review Questions**

1) The absolute values given are for points $A$ and $E$; students may have chosen other points instead, so those answers are also acceptable.

# Operations on Complex Numbers

*Quadratic Formula*

**In-Text Examples**

Sign errors are very likely to occur when working with the quadratic formula.
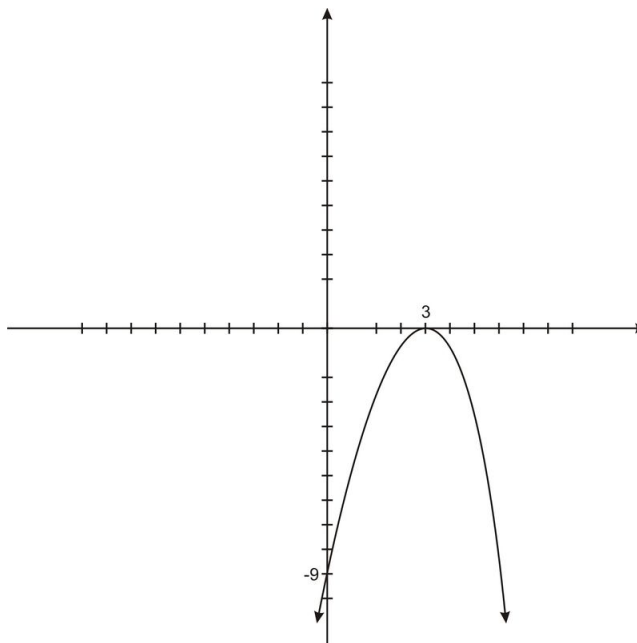
**Review Questions**

1) Students may forget that one side of the equation must equal zero before they can apply the quadratic formula, so they may try to simply read off the coefficients from the left-hand side of the equation.

**Additional Problems**

1) a) What does the discriminant of $-x^2 + 6x - 9$ tell you about its root(s)?

b) Calculate the root(s).

c) Graph the equation.

**Answers to Additional Problems**

1) a) The discriminant is zero, so the function has one repeated real root.

b) The one root is 3.

c)



*Sums and Differences of Complex Numbers*

**Review Questions**

1) The answers students get from adding the vectors graphically may not be as precise as the answers they get from adding the numbers algebraically, so they may not quite match each other. This is fine as long as they are reasonably close.

*Products and Quotients of Complex Numbers (conjugates)*

**In-Text Examples**

1) This is another likely place for sign errors: when multiplying complex numbers and simplifying the answer, students are quite likely to convert $i^2$ to 1 instead of 1, and so simply drop the $i^2$ altogether without changing the sign of its coefficient.

# Trigonometric Form of Complex Numbers

*Trigonometric Form of Complex Numbers: Steps for Conversion*

**In-Text Examples**

3) After learning to work with square roots of negative numbers, students may get a bit confused when finding $r$

*2.6. POLAR EQUATIONS AND COMPLEX NUMBERS*

based on negative values of $x$ and $y$; they might forget that squaring those negative values should yield positive numbers, and that they should not be trying to take the square roots of any negative numbers to find $r$.

**Review Questions**

2) Graphing the number on a rectangular coordinate graph is one option students may take, although that requires expressing it in standard form first and is slightly harder than graphing it in polar form.

**Additional Problems**

1) Express the sum of $3 + 6i$ and $9 - 2i$ in polar form.

**Answers to Additional Problems**

1) The sum of the two numbers is $12 + 4i$, so in rectangular form $x = 12$ and $y = 4$.

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{12^2 + 4^2} \\
&= \sqrt{144 + 16} \\
&= \sqrt{160} \\
&= 4\sqrt{10}
\end{aligned}
$$

$$
\begin{aligned}
\tan\theta &= \frac{y}{x} \\
&= \frac{4}{12} \\
&= \frac{1}{3} \\
\theta &= \tan^{-1}\frac{1}{3} \\
&\approx 18.43°
\end{aligned}
$$

So in polar form the number is $4\sqrt{10}$ cis $18.43°$.

# Product and Quotient Theorems

*Using the Quotient and Product Theorem*

**In-Text Examples**

1) When applying the product rule, probably the easiest error for students to make is to get confused about whether to add the $r-$values and multiply the $\theta-$values or vice versa. Students may also try to add them both or multiply them both.

(Similar errors will occur when applying the quotient rule.)

3) Students may get confused when the numbers to be multiplied or divided are presented in rectangular rather than polar form; they may try to treat the real parts like r-coordinates and the imaginary parts like $\theta-$coordinates, rather than converting to polar form first so they can do the calculations properly.

**Additional Problems**

1) Find the quotient: $(7 + 2i\sqrt{2}) \div (\sqrt{3} - 4i)$. Express your answer in rectangular form rounded to two decimal places.

**Answers to Additional Problems**

1) First, convert to polar form:

$$
\begin{aligned}
r_1 &= \sqrt{x^2 + y^2} \\
&= \sqrt{7^2 + (2\sqrt{2})^2} \\
&= \sqrt{4 + 8} \\
&= \sqrt{57}
\end{aligned}
\qquad\qquad
\begin{aligned}
r_2 &= \sqrt{x^2 + y^2} \\
&= \sqrt{\sqrt{3}^2 + (-4)^2} \\
&= \sqrt{3 + 16} \\
&= \sqrt{19}
\end{aligned}
$$

$$
\frac{r_1}{r_2} = \frac{\sqrt{57}}{\sqrt{19}} = \frac{\sqrt{19.3}}{\sqrt{19}} = \frac{\sqrt{19}\sqrt{3}}{\sqrt{19}} = \sqrt{3}
$$

$$
\begin{aligned}
\theta_1 &= \tan^{-1}\frac{y}{x} \\
&= \tan^{-1}\frac{2\sqrt{2}}{7} \\
&\approx 22.002^\circ
\end{aligned}
\qquad\qquad
\begin{aligned}
\theta_0 &= \tan^{-1}\frac{y}{x} \\
&= \tan^{-1}\frac{4}{\sqrt{3}} \\
&\approx 66.587^\circ
\end{aligned}
$$

$$
\theta_1 - \theta_2 \approx -44.585^\circ
$$

So the quotient in polar form is $\sqrt{3}\operatorname{cis} - 44.585^\circ$.

Converting back to rectangular form:

$$
\begin{aligned}
x &= \sqrt{3}\cos(-44.585^\circ) \\
&\approx 1.23
\end{aligned}
\qquad\qquad
\begin{aligned}
y &= \sqrt{3}\sin(-44.585^\circ) \\
&\approx -1.22
\end{aligned}
$$

So the final answer is $1.23 - 1.22i$.

## Powers and Roots of Complex Numbers

### De Moivre's Theorem

**In-Text Examples**

1) As with the product theorem, students may get confused about which coordinate to multiply by $n$ and which one to raise to the $n^{\text{th}}$ power when applying De Moivre's Theorem. Also, even after writing down [U+0080][U+009C] $\cos n\theta$ [U+0080][U and [U+0080][U+009C] $\sin n\theta$, [U+0080][U+009D] they may still try to find $n\cos\theta$ and $n\sin\theta$ instead.

**Review Questions**

*2.6. POLAR EQUATIONS AND COMPLEX NUMBERS*

1) As with the product theorem once more, students may forget to convert to polar form before applying De Moivre's Theorem…

2) …or may forget to apply De Moivre's Theorem before converting to rectangular form.

### *nth Root Theorem*

**Additional Problems**

1) Find all the fourth roots of 81.

2) Find all the sixth roots of $64i$.

**Answers to Additional Problems**

1) $3, 3i, -3, -3i$

2) The principal root is $2 \operatorname{cis} \frac{\pi}{6}$, or $2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$; the five other roots are $2 \operatorname{cis} \frac{\pi}{2}, 2 \operatorname{cis} \frac{5\pi}{6}, 2 \operatorname{cis} \frac{7\pi}{6}, 2 \operatorname{cis} \frac{3\pi}{2}$, and $2 \operatorname{cis} \frac{11\pi}{6}$.

### *Solve Equations*

**Review Questions**

1) This problem is an easy place to make sign errors.

### *Applications, Trigonometric Tools: Powers and Roots of Complex Numbers*

**In-Text Examples**

1) Students may actually be able to skip the step of calculating the three roots if they realize that the roots are evenly spaced about a circle and are able to figure out how to graph them based on that knowledge. This probably shouldn't be penalized, as it demonstrates understanding of the principles behind $n^{\text{th}}$ roots.

2) The bit about using the Pythagorean Theorem and polar coordinates is somewhat of a red herring; those can be used to find other values in the given diagram, but to find L, students need only use the Law of Cosines.